Why we have cross product only in dimension 3?

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As we have quantities in physics, which have only magnitude, there are also quantites, which have also one or more directions. We can call them vectors or generally tensors. An example could be magnetic permeability:

$$\vec{B} = \mu\vec{H}$$

where $\mu = 4 * 10^{-7} H/m$. So, in free space are $\vec{B}$ and $\vec{H}$ colinear. However, there are exotic materials, where $\bar{\mu}$ is not a scalar, but tensor with two directions in every point. Then $\vec{B}$ and $\vec{H}$ is no longer colinear.

Tensors are defned in mathematics as multilinear mappings

$$T(\overbrace{V, V, ..., V}, \overbrace{V^*, V^*, ..., V^*}) \to R,$$

where we have r copies of n-dimensional vector sprace V and s copies of its dual V^*. [1]

So, we could have tensors of type (r, s) with r contravectors (vectors) and s covectors (forms). We will work with tensors of type $(0, r)$. Antisymmetric tensor $(0, 2)$ is a tensor with the property $T(\vec{u}, \vec{v}) = -T(\vec{v}, \vec{u})$. And we could define antisymmetrization by the following prescription:

$$T_A(\vec{u}, \vec{v}) = \frac{1}{2!}(T(\vec{u}, \vec{v}) - T(\vec{u}, \vec{v}))$$

Completely antisymmetric tensor is a tensor, which is antisymmetric in every 2 arguments. When it is of type $(0, p)$, we call it a p - form. A p-form has at most $\binom{n}{p}$ independent components.[2] We could also define a q -vector, which has also at most $\binom{n}{q}$ independent components, in similar way.

We can define a wedge product by the following definition from two 1 - forms $\widetilde{p}$ and $\widetilde{q}$:

$$\widetilde{p} \wedge \widetilde{q} = \widetilde{p} \otimes \widetilde{q} - \widetilde{q} \otimes \widetilde{p}$$

The key observation is now the following: let's have a q-vector T, $T^{i...k} = T^{[i...k]}$ and dual-form $\tilde{w}$, which maps q-vectors to $(n - q)$-forms by the equation

[1] We could choose a basis in every point of a vector space V. Then we could take an orthonormal vector to every element of the basis and we obtain a reciprocal basis.

[2] The algebra of all p-forms in n-dimensional vector space V is called Grassmann algebra and it has 2^n elements.

$$A_{j..l} = \frac{1}{q!} w_{i..kj..l} T^{i...k}.$$

This is symbolically

$$\tilde{A} = *T.$$

We could write the star of cross product like a wedge product of two vectors:

$$*(\vec{a} \times \vec{b}) = \vec{a} \wedge \vec{b}$$

but this is possible only in dimension 3 !! (We identify vectors and forms in dimension 3!)

Literature

[1] B.F.SCHUTZ: *Geometrical methods of mathematical physics*, Cambridge University Press, 1980