Variational principles and the connection of mathematics and physics

It is a well known fact that we could derive basic equations in physics from variation of action. Action is the time integral of Lagrangian. However, what is the variation mathematically?

We need to define so called directional derivative. Let's have a function from one Hilbert space X to an other Hilbert space Y,

$$f(x) : X \to Y.$$

We define the following limit

$$D_h f(x) = \lim_{\lambda \to 0} \frac{f(x + \lambda h) - f(x)}{\lambda}, \tag{1}$$

and we call it a directional derivative in the direction h. If exists a directional derivative in every direction h and it is a bounded linear functional, we say that the function have a Gâteaux derivative

$$df(x) : h \mapsto D_h f(x).$$

Let's define now the physical action

$$S[x(t)] \equiv \int_{t_1}^{t_2} L(x(t), \dot{x}(t), t)\ dt,$$

which is a time integral from a Lagrange function. We could translate the physical jargon that $\delta S[x(t)] = 0$ by $dS(x) = 0$ and we could derive the equations of motion.(We must only take care about boundary conditions for variations: $h(t_1) = h(t_2) = 0$)

We have from definition that $\frac{d}{d\lambda} S(x + \lambda h)|_{\lambda=0} = D_h S(x)$ and we have the equality

$$D_h S(x) = < \frac{\delta S}{\delta x}, h >,$$

where $\frac{\delta S}{\delta x}$ is the variational derivative, which is the representant of the standard scalar product. This is also useful to know in physical computations.