# Second order action

**Abstract.** We derive the second order action for gravitational waves from inflation.

**Keywords.** Einstein-Hilbert action

The action governing gravitational waves in the second order expansion of the full action is

$$S = S_{EH} + S_{\phi}, \tag{1}$$

where

$$S_{EH} = \frac{M_{Pl}^2}{2} \int \sqrt{-g}\ R\ d^4x \tag{2}$$

and the matter action is

$$S_{\phi} = \int \sqrt{-g}\ L_{\phi}\ d^4x \tag{3}$$

with the scalar-field Lagrangian

$$L_{\phi} = -\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - V(\phi). \tag{4}$$

The perturbed flat FLRW metric $g_{\mu\nu}$ is conformally equivalent to the perturbed Minkowski metric

$$g_{\mu\nu} = a(\eta)^2\bar{g}_{\mu\nu} = a(\eta)^2(\eta_{\mu\nu} + h_{\mu\nu}) \tag{5}$$

where the pure tensor perturbation $h_{\mu\nu}$ is spatial $h_{0\mu} \equiv 0$, traceless $\eta^{\mu\nu}h_{\mu\nu} = 0$ and transverse $\bar{\partial}_{\mu}h^{\mu\nu} = 0$.

The perturbed inverse Minkowski metric is

$$\bar{g}^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2), \tag{6}$$

and the perturbed inverse FLRW metric is $g^{\mu\nu} = a^{-2}\bar{g}^{\mu\nu}$.

For conformally equivalent metrices $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, the associated Ricci scalars are related by

$$\tilde{R} = \Omega^{-2}(R - 6\nabla^2 \ln\Omega - 6\nabla_a \ln\Omega\nabla^a \ln\Omega). \tag{7}$$

Using the formula

$$\det(X+\epsilon A) = \det X \det(I+\epsilon B) = \det X (1+\epsilon\; trB + \frac{\epsilon^2}{2}[(trB)^2 - tr(B^2)]) + O(\epsilon^3),$$
$$B = X^{-1}A. \tag{8}$$

we have the peturbed FLRW metric determinant up to second order

$$\begin{aligned} g^{(0)} &= -a^8, \\ g^{(1)} &= 0, \\ g^{(2)} &= \frac{a^8}{2} h_{\mu\nu} h^{\mu\nu}. \end{aligned} \tag{9}$$

The perturbed Minkowski metric up to second order are

$$\begin{aligned} \bar{\Gamma}^0_{ij} &= \frac{1}{2}\dot{h}_{ij}, \\ \bar{\Gamma}^i_{j0} &= \frac{1}{2}(\dot{h}^i{}_j - h^{ik}\dot{h}_{kj}) + O(h^3), \\ \bar{\Gamma}^i_{jk} &= \frac{1}{2}(\bar{\partial}_j h^i{}_k + \bar{\partial}_k h^i{}_j - \bar{\partial}^i h_{jk} - h^{il}\bar{\partial}_j h_{lk} - h^{il}\bar{\partial}_k h_{lj} + h^{il}\bar{\partial}_l h_{jk}) + O(h^3) \end{aligned} \tag{10}$$

and the others are identically zero to all orders. Then the perturbed Minkowski metric Ricci scalar up to second order is

$$\bar{R} = -h^{ij}\ddot{h}_{ij} + \frac{3}{4}\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} - \frac{3}{4}\dot{h}_{ij}\dot{h}^{ij} + h^{ij}\bar{\partial}_k\bar{\partial}^k h_{ij} - \frac{1}{2}\bar{\partial}^l h^{kj}\bar{\partial}_k h_{lj}. \tag{11}$$

After integration by parts we arrive to

$$S^{(2)}_{EH} = \frac{1}{8}\int a^2 \dot{h}^{ij}\dot{h}_{ij}\; d^4x - \frac{1}{8}\int a^2 \bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} - \frac{1}{4}\int a^2 (H^2 + 2\dot{H}) h^{\rho\sigma} h_{\rho\sigma}. \tag{12}$$

When we compute the matter action we obtain the final result:

$$S^{(2)} = \frac{M_{Pl}}{8}\int a^2(\dot{h}^{ij}\dot{h}_{ij} - \bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk}). \tag{13}$$

**Reference**

[1] Mike S.Wang, Second order action for gravitational waves from inflation