1. Banach space

Definition 1.1. Norm is a nonnegative function $\|v\|$ from vector space $V_{\mathbb{C}}$ (over complex numbers) to real numbers $\mathbb{R}$, $\|.\| : V_{\mathbb{C}} \to \mathbb{R}$, such that the following properties are fulfilled:

- $\|a\| \geq 0$, $\|a\| = 0 \Leftrightarrow a = 0$
- $\lambda \in \mathbb{C}$, $\|\lambda a\| = |\lambda| \, \|a\|$
- $\|a + b\| \leq \|a\| + \|b\|$

Definition 1.2. Metric is non-negative function from $V \times V$ to $\mathbb{R}$, which must fulfill the following:

- $\rho(v, w) \geq 0$ and $\rho(x, y) = 0 \Leftrightarrow x = y$
- $\rho(\lambda v, \lambda w) = \lambda \rho(v, w)$, $\rho(v, w) = \rho(w, v)$
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

Exercise 1.3. Norm is uniformly continuous function.

We have immediately that $\rho(x, y) = \|x - y\|$ is a metric.

Definition 1.4. We will say that vector space X with a norm $\|.\|$, is a Banach space, when the metric space (X, ρ) with the metric ρ induced by this norm $\|.\|$ is complete.

Example 1.5. Continuous functions $f(x) \in C(K)$ on a compact space K form with the maximum norm, $\sup_{x \in K}(|f(x)| : x \in K)$, Banach space.

Example 1.6. Space of all sequences $\{x_n\}_{n \in \mathbb{N}}$, such that $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ is finite, form for given $p \geq 1$ Banach space.

Definition 1.7. We say that a map $(.,.) : V \times V \to \mathbb{R}$ (or $\mathbb{C}$) is a scalar product when it fulfills:

- $(a, a) \geq 0$, and $(a, a) = 0 \Leftrightarrow a = 0$
- $(a, \lambda b) = \bar{\lambda}(a, b)$, $(\lambda a, b) = \lambda(a, b)$, $(a, b) = \overline{(b, a)}$
- $(a, b + c) = (a, b) + (a, c)$

Definition 1.8. We say that a Banach space H is a Hilbert space, when the metric space (H, ρ) is complete in the metric induced by this $\|a\| = \sqrt{(a, a)}$.

Example 1.9. The space of square integrable functions $f : \mathbb{R} \to \mathbb{R}$ with a norm $\sqrt{\int_{\mathbb{R}} |f|^2 \, dx} < \infty$ is Hilbert space.

Example 1.10. The space of sequences x_n with the norm defined in (1.6), $p = 2$, is Hilbert space.

We have two basic notions of a basis in infinite dimensional Hilbert spaces. We have orthonormal basis and algebraical Hamel basis.

Definition 1.11. We say that system of vectors L creates orthonormal system, when all vectors from this system are perpendicular to each other and they have norm 1. System is a basis, when we could not add any further functions to this system.

Corollary 1.12. *Every Hilbert space has an orthonormal basis.*

Proof. Could be done by Zorn's lemma. □

Definition 1.13. Topological space is separable, when there exists countable dense set.

Corollary 1.14. *Hilbert space is separable, if and only if there exists countable orthonormal basis.*

Proof. If there is uncountable orthonormal basis, then the space is not separable, because $\|e_k - e_n\| = \sqrt{2}$ for the elements of the basis e_k and e_n. □

2. Bounded operator

We will develop now the concept of bounded linear operators. Bounded operator maps bounded sets on bounded sets. The following lemma holds:

Lemma 2.1. *The following is equivalent for linear mapping $A : X \to Y$, where X and Y are normed vector spaces :*

- *A is continuous*
- *A is continuous in 0*
- *A is bounded on B_X*
- *A is bounded*

Proof. When A is continuous then is continuous in 0.

If A is continuous in 0, then for all $\epsilon > 0$ exists a $\delta > 0$ such that

$$\|x\| < \delta \Rightarrow \|A(x)\| < \epsilon.$$

There exists $\delta_0 > 0$, such that $\|A(x)\| \leq 1$, when $x < \delta_0$. But then for every $z \in B_m$ holds $\|A(z)\| = \frac{1}{\delta_0} \|A(\delta_0 z)\| \leq \frac{1}{\delta_0}$.

Let $\|Ax\| \leq K$ for $x \in B_m$, then there is $M < \infty$: $\|A(x)\| \leq M \|x\|$ for every $x \in X$; Let $\epsilon > 0$, if $x, y \in X$: $\|x - y\| < \frac{\epsilon}{M}$, then $\|Ax - Ay\| \leq M \|x - y\| < \epsilon$. Then A is also uniformly continuous. □

Definition 2.2. We define the norm of linear mapping $L : X \to Y$ like $\|L\| = \sup_{\|x\| \leq 1} \{\|Lx\|_Y : \|x\|_X \leq 1\}$

Exercise 2.3. When we consider the space of all linear mappings from normed linear space X to normed linear space Y, $\Lambda(X, Y)$, with the norm defined in (2.2), it is a Banach space when Y is a Banach space.

We say that for x and y in a Hilbert space H, $x \perp y$, when $(x, y) = 0$. We define similarly $x \perp A$, where A is subspace of H and $A \perp B$, where A, B are subspaces of H.

Definition 2.4. We define orthogonal complement of subspace $M \subset\subset H$ like the set of vectors $x \in M^{\perp}$, such that $(x, h) = 0$ for all $h \in M$.

Lemma 2.5. *Let M be closed subspace of Hilbert space H. Then for every $x \in H$ exists exactly one $m_0 \in M$, $\|x - m_0\| = dist(x, M)$.*

Proof. Let's suppose that $x \neq 0$. Otherwise we will put $m_0 = 0$. The task is to prove existence and uniqueness of an element of the set $C = x - M$, such that the norm of this element, $\delta = dist(0, C) = dist(x, M)$, has minimal value. We will prove the existence: the first observation is $0 \notin C$, so $\delta > 0$; According to the definition of distance, there exists a sequence $y_n \in C$ such that $\|y_n\| \to \delta$. If we prove that the sequence is Cauchy then there will exist y such that $\|y\| = \lim \|y_n\| = \delta$.

Sequence $\frac{1}{2}(y_n + y_k) \in C$. Therefore we have by rectangular rule that

$$\|y_n - y_k\|^2 = 2(\|y_n\|^2 + \|y_k\|^2) - \|y_n + y_k\|^2 \leq 2(\|y_n\|^2 + \|y_k\|^2) - 4\delta^2 \to 0$$

The proof of uniqueness is also straightforward: let $a \in C$ and $b \in C$ are 2 vectors, such that we obtain the minimum value $dist(0, C)$ for both of them. Then again by rectangular rule

$$\|a - b\|^2 = 2(\|a\|^2 + \|b\|^2) - \|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2) - 4\delta^2 \to 0$$

□

We need also some geometrical characterization.

Lemma 2.6. *Let M is a subspace of Hilbert space H. Then when $x \in H$ and $m_0 \in M$, we have the following equivalence: $\|x - m_0\| = dist(x, M)$ if and only if $x - m_0 \perp M$.*

Proof. Let $x - m_0 \in M^\perp$ and $m \in M$. Then

$$\|x - m\|^2 = \|x - m_0\|^2 + \|m - m_0\|^2 \geq \|x - m_0\|^2$$

So, we proved by Pythagoras theorem one implication, because $\|x - m_0\| = dist(x, M)$.

For the second implication:

$$(x - m_0, x - m_0) \leq \|x - (m_0 + \epsilon m)\| = \|x - m_0\|^2 - \epsilon(x - m_0, m) + \epsilon^2 \|m\|^2$$

Because we have chosen $\epsilon > 0$ arbitrary, we have $(x - m_0, m) \leq 0$. When we choose $\epsilon < 0$, we get opposite inequality. This means $(x - m_0, m) = 0$. □

Definition 2.7. We say that vector space E is an algebraical sum of spaces M and N, $E = M \oplus N$, when every vector $v \in E$ could be uniquely written as $v = v_M + v_N$, where $v_M \in M$ and $v_N \in N$. We have $E = M \oplus N$ and $M \cap N = \emptyset$.

So, if $E = M \oplus N$, then we can define projections $P_M(v) = v_M$ and $P_N(v) = v_N$. When E has also topological structure, we can define the following notion:

Definition 2.8. If $W = M \oplus N$, we say that W is a topological sum of M and N, $W = M \oplus_t N$, if projections P_M and P_N are continuous.

We immediately see that P_M is continuous if P_N is continuous. If W is topological sum of subspaces M and N , then both spaces M and N are closed. Contrary, if $W = M \oplus N$ and both spaces M and N are closed, then $W = M \oplus_t N$.

Definition 2.9. We say that a continuous and linear operator P on Banach space X, $P : X \to X$ is a projection, if $P^2 = P$.

If P is a projection, then $I - P$ is projection and $\|P\| \geq 1$.

Lemma 2.10. $P : X \to X$*:*

- *If P is a projection on Banach space X, then $\ker(P)$ and $\mathrm{Im}(P)$ are closed subspaces X and $X = \ker(P) \oplus_t \mathrm{Im}(P)$.*
- *If M and N are closed subspaces of X, $X = M \oplus_t N$, then exists P on X such that $M = \ker P$ and $N = \mathrm{Im}\, P$*

Proof. $\ker P$ is always closed subspace. But because $\ker(I-P) = \operatorname{Im} P$, $\operatorname{Im} P$ is also closed subspace. We can write

$$x = Px + (I-P)x,$$

but Px is element of $\operatorname{Im} P$ and $\ker(I-P)x$ is element of $\ker P$. $\ker P \cap \operatorname{Im} P = \emptyset$ and the first result follows.

For the next statement: we can write $x = x_M + x_N$, where we define $Px = x_N$; □

Now we define the notion of orthogonal projections (or projectors) in Hilbert spaces:

Definition 2.11. Let M is a closed subspace of a Hilbert space H. We define projector P like the mapping, which projects orthogonally any $v \in H$ to M, $Pv \in M$.

We see that projector P is a linear mapping and it is bounded. We see that $\operatorname{Im} P = M$ and that $\ker P = M^\perp$ and $\|P\| = 1$. $I - P$ is projector on $M^\perp$.

Definition 2.12. We say that an operator A defined on Hilbert space H is Hermitian, if $(Ax, y) = (x, Ay)$ for all $x, y \in H$.

Lemma 2.13. *Operator A, defined on H is a projector, if and only if it is Hermitian and $A^2 = A$. If these conditions are fulfilled then $\operatorname{Im}(A)$ is a closed subspace and it is composed from such elements $x \in H$ that $Ax = x$.*

Proof. When A is a projector, then $A^2 = A$ is from definition and it is Hermitian, because $(Ax, (I-A)x) = 0$. And so $(Ax, x) = \|Ax\|^2 \in \mathbb{R}$.

Contrary, when it is Hermitian and $A^2 = A$. Let's take $y = \lim Ax_n$, then $Ay = \lim A^2 x_n = y$. So, $\operatorname{Im}(A)$ is closed and for its elements holds that $Ey = y$. When we write $y \equiv Ax$ and $z \equiv (I-A)x$, then $(y, z) = 0$, so A is really orthogonal projection.

□

Every projector is projection and it is positive operator.

Definition 2.14. Projector E is orthogonal to projector F, if $\operatorname{Im}(E) \perp \operatorname{Im}(F)$. This is equivalent to $EF = FE = 0$.

Lemma 2.15. *Let E, F are projectors. Then*

(1) *$E + F$ is a projector, if E is orthogonal to F; Then*

$$\operatorname{Im}(E + F) = \operatorname{Im}(E) \oplus \operatorname{Im}(F)$$

(2) *Following is equivalent:*
- *$E - F$ is a projector*
- *$E \geq F$*
- *$\operatorname{Im} E \supset \operatorname{Im} F$*
- *$EF = FE = F$*

Then $\operatorname{Im}(E - F)$ is orthogonal complement of subspace $\operatorname{Im} F$ in $\operatorname{Im} E$.

(3) *Operator EF is a projector, iff $EF = FE$. Then $\operatorname{Im}(EF) = \operatorname{Im}(E) \cap \operatorname{Im}(F)$.*

Proof. When $EF = FE = 0$, then we have directly from the definition of a projector that $E+F$ is projector. Opposite implication: when $EF+FE = 0$, we will multiply this equation by F from left and right and we obtain desired equality $EF = FE = 0$.

When $E-F$ is a projector, then $((E-F)x,x)\geq 0$ implies $(Ex,x)\geq(Fx,x)$. The first inequality holds because $E-F$ is projector and so it is positive.

From the inequality $E\geq F$ follows: $\|Ex\|\geq\|Fx\|$; But we know that $\operatorname{Im}E$ is the set of elements such that $Ey=y$. Because $\|Ex\|\leq\|x\|$ for every projector, we can characterize the set of elements $y\in\operatorname{Im}E$ like $\|Ey\|=\|y\|$. But when $\|x\|=\|Fx\|$, then $\|Ex\|\geq\|Fx\|=\|x\|$. Then we have $\|x\|=\|Ex\|$.

If $\operatorname{Im}E\supset\operatorname{Im}F$, then $\ker E\subset\ker F$, but $FE=F$ implies $F(I-E)=0$.

If EF is projector, then $EFEF=FE$ and $(EFx,x)=(x,EFx)$. But then also $(EFx,x)=(x,FEx)$ and the result follows. □