1. Hahn-Banach theorem and applications

Definition 1.1. We say that p is a convex functional, if:

- $p(\lambda x) = \lambda p(x)$, for $\lambda \geq 0$.
- $p(x+y) \leq p(x) + p(y)$

We say that p is a pseudonorm, if $p(\lambda x) = |\lambda| p(x)$.

First we will write algebraical Hahn-Banach theorem:

Theorem 1.2. *Let W be a vector space and p is a convex functional on this space. Let $M \subset\subset W$ be normed linear subspace of space W. f is a functional on M, such that $f \leq p$ on M. Then there exists a F, which is equal to f on M, such that $F \leq p$ on W.*

When p is an pseudonorm, we can choose F such, that $|F| \leq p$ in whole W.

From this theorem we will prove the standard Hahn-Banach theorem:

Theorem 1.3. *Let E is a normed linear space with subspace M, $M \subset\subset E$. Let be f a functional on M. Then there exists a functional F on E, such that $F = f$ on M and $\|F\|_E = \|f\|_M$.*

Proof. We can put $p(x) \equiv \|f\|_M \|x\|_E$. □

We have 2 basic results, which follow from this theorem:

Lemma 1.4. *Let E be a normed linear space, where x is a non-zero vector, $x \neq 0$. Then there exists a non-zero functional f, such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Proof. The proof is immediate. We can choose $M \equiv \{\lambda x, \lambda \in \mathbb{R}\}$ and define $f(\lambda x) = \lambda \|x\|$. □

Lemma 1.5. *Let E be normed linear space, where M is a closed subspace $M \subset\subset E$ and v_0 is a vector, which is not contained in M. Then it exists a functional, which is zero on M and $f(v_0) \neq 0$.*

We will need also following 2 results:

Lemma 1.6. *Let f be linear form on normed linear space M. Then f is continuous, if and only if Ker(f) is closed subspace.*

Proof. When f is continuous, then Ker$f$ is a closed subspace.

Contrary, if Ker$f$ is a closed subspace. Then choose z such, that $f(z) = 1$. Because Ker$f$ is closed, we can find $\delta > 0$ such that $z + U \cap \text{Ker} f = \emptyset$, $U \equiv \{x : \|x\| < \delta\}$. But then we will find that f is bounded on some neighberhood of 0. □

Lemma 1.7. *Let G be an open set in normed linear space E. If f is continuous functional on E, then $f(G)$ is open.*

Proof. Immediate. But it is useful to compare this theorem with open mapping theorem. □

Let's define hyperplanes:

Definition 1.8. We say that M is a hyperplane in normed linear space E, if it exists a functional f and real number α, such that $M = \{x \in E : f(x) = \alpha\}$. We can similarly say that M is a hyperplane, if $M = x + W$, where W is maximal nonzero subspace of E.

We say that M is closed, when $M = x + W$, where W is maximal closed nonzero subspace of E.

Now we will write, so called Mazur theorem:

Theorem 1.9. *If C is an open convex set in normed linear space E and x is a vector, which is not contained in E, $x \notin C$. Then it exists a closed hyperplane H, such that $x \in H$ and $H \cap C = \emptyset$.*

Proof. We can suppose that $x \notin C$, because otherwise we can shift x to zero. Let's put now $G \equiv \cup\{\lambda C : \lambda > 0\}$. Then G is open, it is the smallest cone which contains C. Let's now define

$$p(x) \equiv \inf\{\|x + y\|, y \in G\}$$

for $x \in E$. Then $p(x)$ is a convex functional and $p(x) > 0$ on G.

Now according to the algebraical form of Hahn-Banach theorem exists a linear form f such that $f \leq p$ on E. We see that $f \geq 0$ on G and if $f(z) = 0$ for some $z \in G$, then $f(z_0) = 0$ for all z_0 in some open neighberhood of this point z and so $f = 0$ on the whole E.

Put $H = \mathrm{Ker} f$. Then $0 \in H$ and $H \cap G = \emptyset$. We need to prove that f is continuous.

□

Finally we will write geometrical Hahn-Banach theorem:

Theorem 1.10. *If A and B are 2 disjoint open convex sets (or one closed and second compact) in normed linear (real) space E. Then there exists a functional f, such that $A \subset \{x \in E : f(x) > \alpha\}$ and $B \subset \{x \in E : f(x) < \alpha\}$.*

Proof. Let's consider the following set: $C \equiv A - B$; C does not contain 0 and because

$$C = \sum_b \{A - b\},$$

we see that it is convex and open. It exists a mapping f such that $f > 0$ according to the Mazur theorem. Then when $a \in A$ and $b \in B$, we have $f(a) - f(b) > 0$. It exists an α, such that $\sup\{f(b) : b \in B\} \leq \alpha \leq \inf\{f(a) : a \in A\}$. We have that $f > \alpha$ on A and $f < \alpha$ on B, because A and B are open. □

Now we will state a useful lemma, which could be used as a definition of norm:

Lemma 1.11. *Let $x \in E$. Then $\|x\| = max\{|\varphi(x)| : \varphi \in E^*, \|\varphi\| \leq 1\}$.*

Proof. Follows from Lemma 1.4. □

Lemma 1.12. *Let $M \subset\subset E$. If the only functional, which is zero on M is a zero functional, then M is dense in E.*

Proof. It follows from Lemma 1.5. □

Lemma 1.13. *If $M \subset\subset E$ is a finite dimensional subspace of a Banach space X. Then it has a topological complement.*

Proof. Let $e_1, e_2, ..., e_n$ be a basis of M. Then define M_i like a linear span of $\{e_j : e_j \neq e_i\}$. Then M_i is closed and according to Lemma 1.5 we can define a functional φ such that $\varphi(e_i) = 1$ and $\varphi(M_i) = 0$. Then the following operator is a projection on M:

$$P(x) = \sum_i \varphi(x) e_i \tag{1}$$

□