## 1. Riesz-Schaudauer theory of compact operators

We will denote the range of an operator $T$, $T \in L(X)$, like $R(T)$. It is always a subspace of space $X$. It is closed in the case of finite dimensional space $X$.

**Lemma 1.1.** *Let be $X$ a Banach space and $T \in L(x)$. If it exists a $\beta > 0$ such, that $\|T(x)\| \geq \beta \|x\|$ for all $x \in X$, then $R(T)$ is a closed set.*

*Proof.* Let is $z_n \in R(T)$, $z_n \to z$. We will find $x_n \in X$ such that, $T(x_n) = z_n$. Then $\|x_n - x_k\| \leq \beta^{-1} \|z_n - z_k\|$ and we see that a sequence $\{x_n\}$ is Cauchy. Then the limit $\lim x_n \equiv x$ exists and we obtain from the continuity of $T$ that $Tx = z$. □

We will call a complex number an eigenvalue of an operator $T \in L(x)$, when exists $x \neq 0$, for which $T(x) = \lambda x$. So, $\lambda$ is an eigenvalue of $T$ , when the operator $T - \lambda I$ is not injective. We denote the set of all eigenvalues of $T$ like $\sigma_p(T)$ and we call it pointwise specturm of $T$. [1]

We will say that a complex number $\lambda \in \mathbb{C}$ lies in spectrum of $\sigma(T)$ of operator $T$, when operator $T - \lambda I$ is not injective , or $R(T - \lambda I) \neq X$. It is clear that in the spectrum of operator $T$ lie all its eigenvalues.

If $\lambda$ is not in spectrum of operator $T$, the operator $T - \lambda I$ is injective and his range is whole space $X$. We can characterize operators with such a property also followingly: we say that an operator $T$ defined on Banach space $X$ is invertible, if exists an operator $L \in L(X)$ such, that $LT = TL = I$.

**Theorem 1.2.** *Operator $T \in L(X)$ is invertible, iff $T$ is injective and onto.*

*Proof.* If $T$ is invertible. Then there exists $L \in L(X)$ such, that $LT = I$ and $TL = I$. If $Tx = 0$, then we get from the first equality that $x = 0$. So $T$ is injective. If we choose $y \in X$ arbitrarily, we can put $x = Ly$ and we obtain from the second inequality that $y = Tx$. So $T$ is onto.

If we have an injective operator $T \in L(X)$ with the property that $R(T) = X$, then the inverse mapping (in the set-theoretic sense) is a linear operator onto $X$, which is according to Banach open mapping theorem continuous. We have

$$T(T^{-1}x) = T^{-1}(Tx) = x,$$

for all $x$. □

**Lemma 1.3.** *Let $T$ be an invertible operator on Banach space $X$, $\alpha \equiv \left\|T^{-1}\right\|$. If $S \in L(X)$ and $\|S - T\| < \frac{1}{\alpha}$, then operator $S$ is invertible.*

*Proof.* Because

$$\left\|\sum_{j=k}^{n}(T^{-1}(T-S))^j\right\| \leq \sum_{j=k}^{n} \left\|(T^{-1}(T-S))^j\right\| \leq \sum_{j=k}^{n} \left\|(T^{-1}(T-S))\right\|^j \leq \sum_{j=k}^{n} q^j$$

,

for all $k < n$, where $q \equiv \alpha \|S - T\| < 1$, then is the sequence of partial sums of serie $\sum_{j=0}^{\infty}(T^{-1}(T-S))^j$ Cauchy in $L(X)$ (this serie is absolutely convergent). So we can put

[1] Contrary to the situation of finite dimensional spaces, there exists operators which are injective but not surjective and which are surjective but not injective.

$$L \equiv \sum_{j=0}^{\infty} (T^{-1}(T-S))^j) T^{-1}.$$

Because $S = T(I - T^{-1}(T-S))$, we obtain by calculation that $LS = SL = I$. □

This lemma tells us the the set of invertible operators on Banach space X is open in $L(X)$.

**Theorem 1.4.** *Let $X$ is a Banach space and $T \in L(X)$. Then the spectrum $\sigma(T)$ is a compact subset of complex plane $\mathbb{C}$. We even have $\sigma(T) \subset \{\lambda \in \mathbb{C}, |\lambda| \leq \|T\|\}$.*

*Proof.* Let's choose $\lambda \in \mathbb{C}$, , $|\lambda| > \|T\|$. Operator $A \equiv -\lambda I$ is invertible. If we put $S \equiv T - \lambda I$, then $\|S - A\| = \|T\| < |\lambda| = \left\|T^{-1}\right\|^{-1}$. Then $S - \lambda I$ is invertible, in other words $\lambda \neq \sigma(T)$. □

Now we will prove Riesz lemma:

**Lemma 1.5.** *Let be $X$ a normed linear space and $Y \subset\subset X$ its closed subspace. Then for every $\epsilon > 0$ exists $x_\epsilon \in X$ such, that*

$$\|x_\epsilon\| = 1, \ dist(x_\epsilon, Y) \geq 1 - \epsilon.$$

*Proof.* Let's suppose that $0 < \epsilon < 1$ and $x \in X \setminus Y$. Because $d \equiv dist(x, Y) > 0$, there exists $x' \in Y$ for which $\|x - x'\| \leq \frac{d}{1-\epsilon}$. If we put $x_\epsilon = \frac{x - x'}{\|x - x'\|}$, then $\|x_\epsilon\| = 1$ and

$$\|z - x_\epsilon\| = \frac{1}{\|x - x'\|} \|(\|x - x'\|)z + x') - x\| \geq \frac{dist(x, Y)}{\|x - x'\|} \geq 1 - \epsilon. \tag{1}$$

□

The following theorem is called Riesz theorem:

**Theorem 1.6.** *Let be $X$ a normed linear space. The following is equivalent:*

- *$X$ is finite dimensional*
- *closed unit sphere $\{x \in X : \|x\| \leq 1\}$ is compact*
- *identical mapping in $X$ is compact*

*Proof.* We need to prove only that if identical mapping in $X$ is compact, then $X$ is finite dimensional: so, let's suppose that $dim\ X$ is infinity; Then exist subspaces $X_1 \subset X_2 \subset ... \subset X$, such that $dim\ X_n = n$. We will find according to the previous Riesz lemma, that there exists a sequence $\{x_n\}$, such that

$$\|x_n\| = 1, \ x_{n+1} \in X_{n+1}, \ dist(x_{n+1}, X_n) \geq \frac{1}{2}.$$

From this follows that $\|x_n - x_m\| \geq \frac{1}{2}$ for $m \neq n$, but then an identical mapping could not be compact. □

For $Ker(T)$ holds always that it is closed, but $R(T)$ hasn't to be closed.

**Lemma 1.7.** *Let be $K$ a compact operator on normed linear space $X$. Then $Ker(I - K)$ is a closed finite dimensional subspace of $X$*

*Proof.* We will denote $B \equiv \{x \in Ker(I - K) : \|x\| \leq 1\}$. Because $Ker(I - K) = \{x \in X : Kx = x\}$, it is $KB = B$ and therefore $B$ must be relatively compact. $Ker(I - K)$ is according to the Riesz theorem finite dimensional. □

**Lemma 1.8.** *If $K$ is a compact operator on infinite-dimensional normed linear space $X$, then $0 \in \sigma(K)$.*

*Proof.* If the $\sigma(K)$ does not contain 0, $K$ is an invertible operator. But then $I = K \circ K^{-1}$ and so $I$ is a compact operator. And this would be a contradiction with Riesz theorem. □

**Lemma 1.9.** *Let be $K$ a compact operator on Banach space $X$. Then $R(I - K)$ is a closed subspace of $X$.*

*Proof.* Let $T = I - K$ and $Z = Ker(I - K)$. Then $Z$ is a closed finite dimensional subspace of $X$. It has a topological complement according to one of the previous lemmas. Let's denote it $W$. Because $W$ is closed, it is a Banach space. Then if we will prove that $T$ is bounded from below, we are done according to Lemma 1.1.

So, suppose that $T$ is not bounded from below. Then there exists a sequence $x_n$ such that $\|x_n\| = 1$ and $Tx_n \to 0$. Because $K$ is compact, we can suppose that $Kx_n \to y$. Then $\lim x_n = \lim(T(x_n) + K(x_n)) = y$. Then $y = 0$, but this is impossible. □

We define annihilators folowingly: let be $X$ a normed linear space, $M \subset\subset X$ and $N \subset\subset X^*$. We define the annihilators by the prescription

$$M^{\perp} \equiv \{\varphi \in X^* : \varphi(x) = 0, \forall x \in M\},$$
$$^{\perp}N \equiv \{x \in X : \varphi(x) = 0, \forall \varphi \in N\}.$$

**Theorem 1.10.** *Let be $X$ and $Y$ normed linear spaces and $T \in L(X, Y)$. We have*

$$Ker(T') = R(T)^{\perp}, \quad Ker(T) =^{\perp} R(T').$$

*Proof.* $a \in Ker(T')$, iff $T'(a)(x) = a(Tx) = 0$ for all $x \in X$. So, $a \in R(T)^{\perp}$.

$x \in Ker(T)$, if $Tx = 0$. But $Tx = 0$, iff $\varphi(Tx) = T'(\varphi)(x) = 0$ for all $\varphi \in X^*$. This means $x \in^{\perp} R(T')$.

□

**Theorem 1.11.** *Let be $X$ and $Y$ normed linear spaces and $T \in L(X, Y)$. Then*

$$\overline{RT} =^{\perp} Ker(T').$$

*Proof.* Let $y \in R(T)$ and $\varphi \in Ker(T')$. So, then there exists $x \in X$ such that $Tx = y$. But then $\varphi(y) = \varphi(Tx) = T'(\varphi)(x) = 0$. But the annihilator is always closed, so we have one inclusion $\overline{RT} \subset^{\perp} Ker(T')$.

We will prove the second inclusion. Let $y \notin \overline{R(T)}$. There exists according to Hahn-Banach theorem $\varphi \in Y^*$, such that $\varphi(y) \neq 0$ and $\varphi(\overline{R(T)} = 0$ . If $x \in X$, then $\varphi(T(x)) = 0$, so $T'(\varphi(x)) = 0$ and $\varphi \in Ker(T')$. But because $\varphi(y) \neq 0$, we have $y \notin^{\perp} Ker(T')$. □

Very important are the following Fredholm alternatives:

**Theorem 1.12.** *Let be $K$ a compact operator on Banach space $X$ and $\lambda \neq 0$. Then the operator $K - \lambda I$ is injective, iff it is surjective.*

**Theorem 1.13.** *Let be $K$ a compact operator and $\lambda \neq 0$. Then*

$$R(K' - \lambda I') = Ker(K - \lambda I)^{\perp}, \quad R(K - \lambda I) =^{\perp} (K' - \lambda I').$$

**Theorem 1.14.** *If it is $K$ a compact operator on Banach space $X$ and $\lambda \neq 0$. Then is*

$$dim\ Ker(K - \lambda I) = dim\ Ker(K' - \lambda I') < 0.$$

## References

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