1. Principle of uniform boundedness

We will first prove the principle of uniform boundedness. We will need for the proof of this theorem the Baire category theorem.

Theorem 1.1. *Let X is a Banach space and E is normed linear space and $G \subset L(X,E)$. The following statements are equivalent:*

- $\sup\{\|L\| : L \in G\} < \infty$
- $\sup\{\|Lx\| : L \in G\} < \infty$ *for every* $x \in X$.

Proof. We need to prove only one implication. Let's put

$$F_n = \cap_{L \in G}\{x \in X : \|Lx\| \leq n\} \text{ for } n \in \mathbb{N}.$$

Sets F_n are closed, because norm is continuous and L are continuous mappings. Because $X = \cup_n F_n$, there exists according to the Baire theorem such k that the set F_k has a nonempty interior. So exists $\epsilon > 0$ and $z \in F_k$ such that $\{x \in X : \|x - z\| < 2\epsilon\} \subset F_k$. Let's fix $L \in G$ and $y \in X, \|y\| \leq 1$. Because $x \equiv z + \epsilon y \in F_k$, we will get

$$\|Ly\| = \left\| L(\frac{x-z}{\epsilon}) \right\| \leq \frac{1}{\epsilon}(\|Lx\| + \|Lz\|) \leq \frac{2k}{\epsilon},$$

and therefore $\|L\| \leq \frac{2k}{\epsilon}$. □

The next result is the Banach - Steinhaus theorem:

Theorem 1.2. *Let X is a Banach space and E is a normed linear space. Let's $L_n \in L(X,E)$ is such a sequence, that exists* $\lim L_n x$ *for every n. We will denote such a limit Lx. Then $L \in L(X,E)$.*

Proof. Direct application of principle of uniform boundedness. □

Next lemma is also a direct application of principle of uniform boundedness:

Lemma 1.3. *Let be $\{x_n\}$ weakly convergent subsequence in normed linear space E. Then the sequence $\|x_n\|$ is strongly bounded.*

Proof. We will choose $\varphi \in E^*$, then $\varphi(x_n)$ is convergent. So, there exists a constant c_φ, which depends only on φ, such that $|\varphi(x_n)| \leq c_\varphi$ for every n. We now define a mapping for every $n \in \mathbb{N}$

$$g_n : \varphi \mapsto \varphi(x_n) \text{ for } \varphi \in E^*,$$

we have $g_n \in E^{**}$ and $\|g_n\| = \|x_n\|$. Because the space E^* is complete and the sequence $\{g_n\}$ is pointlike bounded, then the sequence of norms $\{\|g_n\|\}$ is according to principle of uniform boundedness bounded. □

Lemma 1.4. *We can choose a weakly convergent subsequence from every bounded sequence in reflexive Banach space X.*

Proof. Let x_n is a bounded sequence in a reflexive Banach space X. We can consider a closed linear span of the sequence x_n, which we will denote Y. Then Y is clearly separable and Y^* is also separable. Let φ_n is its dense subset. Because $\varphi_1(x_n)$ is bounded, we can choose a subsequence of x_n, x_n^1, such that $\varphi_1(x_n^1)$ is convergent. And we can further continue, we will choose a subsequence of x_n^1 such that $\varphi_2(x_n^2)$ is convergent, etc. Then finally $\varphi_k(x_n^n)$ converges for each k. Then it is not difficult to show that $\varphi(x_n^n)$ must converge for every $\varphi \in Y^*$. If we choose $\varphi \in Y^*$ arbitrarily and $\epsilon > 0$, then we can choose k such that $\|\varphi - \varphi_k\| < \epsilon$. If we will also denote $K = \sup\{\|x_n\| : n \in \mathbb{N}\}$, then it is according to the assumption $K < \infty$ and

$$|\varphi(x_i^i) - \varphi(x_j^j)| \leq 2K \|\varphi - \varphi_k\| + |\varphi_k(x_i^i) - \varphi_k(x_j^j)| \leq (2K+1)\epsilon$$

for big enough i, j. Because $\varphi(x_n^n)$ is Cauchy, it is convergent. So we can define for $\varphi \in Y^*$

$$F(\varphi) = \lim \varphi(x_n^n).$$

Because $\{x_n\}$ was bounded sequence, there is $F \in Y^{**}$. Because the space Y is reflexive, there exists $x \in Y$ such that $\varphi(x) = F(\varphi)$ for every $\varphi \in Y^*$. Then $x_n^n \xrightarrow{w} x$. □

Every continuous linear operator take weakly convergent sequences to weakly convergent sequences. Compact operator takes weakly convergent sequences to strongly convergent:

Theorem 1.5. *Let be M and N normed linear spaces and T is a compact operator. If $x_n \to x$, then $Tx_n \to Tx$.*

Proof. Let's suppose that $x_n \xrightarrow{w} 0$, then we have $T(x_n) \xrightarrow{w} T(0) = 0$. Let's suppose that sequence $\{T(x_n)\}$ dosn't converge to zero. Then there exists $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ such that $\|T(x_{n_k})\| \geq 0$ for all n_k. According to one of the previous theorems is the whole sequence $\{x_n\}$ bounded. So bounded is also its subsequence $\{x_{n_k}\}$. From the definition of compactness of T exists $y \in N$ and a subsequence $\{y_j\}$ from $\{x_{n_k}\}$, such that $Ty^j \to y$. But because $Ty^j \xrightarrow{w} 0$, then $y = 0$. Therefore $\|Ty_j\| \to 0$ and we get contradiction. □

Definition 1.6. We will say that a mapping $T \in L(X, Y)$ is open, if $T(G)$ is open subset of the range $T(X)$, whenever G is open subset of X.

Theorem 1.7. *Let be X and Y Banach spaces and $T \in L(X, Y)$. If the range of T is Y, then T is an open mapping.*

Proof. We will write only the main idea of the proof. It is sufficient to prove that TV is an open neighborhood of 0, when V is an open sphere with the center in zero. If U is an open set and $y \in TU$, then we find $x \in U$ and $\epsilon > 0$ such, that $Tx = y$ and $x + U(0, \epsilon) \subset U$. Then exists $\delta > 0$ such that $U(0, \delta) \subset TU(0, \epsilon)$. But then

$$y + U(0, \delta) \subset Tx + TU(0, \epsilon) = T(x + U(0, \epsilon)) \subset TU.$$

□

Theorem 1.8. *If X and Y are Banach spaces and T is an injective linear mapping X onto Y (surjective). Then T^{-1} is continuous.*

Proof. The proof follows from open mapping theorem. □

Definition 1.9. We say that a mapping $T : M \to N$, M and N are normed linear spaces, is closed, when the set (x, Tx) is closed in $M \times N$. We could say alternatively that T is closed when

$$z_n \to z \text{ and } T(z_n) \to y$$

implies $Tz = y$.

Every continuous mapping is closed, but every closed mapping hasn't to be continuous.

Theorem 1.10. *Let be T, $T : X \to Y$, closed mapping from Banach space X to Banach space Y. Then it is continuous.*

Proof. Graph of T is a closed subspace of Banach space $X \times Y$. We define the mapping $P : (x, Tx) \mapsto x$. Then the following holds:

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Lx\| = \|(x, Lx)\|$$

Mapping P is linear, injective and surjective. $\|P\| \leq 1$. The inverse mapping P^{-1} is bounded accordiing to closed mapping theorem. Then

$$\|Lx\| \leq \|x\| + \|Lx\| = \|(x, Lx)\| = \left\|P^{-1}(x)\right\| \leq \left\|P^{-1}\right\| \|x\| .$$

□

It follows from this theorem:

Lemma 1.11. *Let a Banach space is an algebraical sum of closed subspaces M and N, $X = M + N$. Then X is also their topological sum.*