1. Principle of uniform boundedness

We will first prove the principle of uniform boundedness. We will need for the proof of this theorem the Baire category theorem.

**Theorem 1.1.** Let $X$ is a Banach space and $E$ is normed linear space and $G \subset L(X, E)$. The following statements are equivalent:

- $\sup\{\|L\| : L \in G\} < \infty$
- $\sup\{\|Lx\| : L \in G\} < \infty$ for every $x \in X$.

**Proof.** We need to prove only one implication. Let’s put $F_n = \cap_{L \in G}\{x \in X : \|Lx\| \leq n\}$ for $n \in \mathbb{N}$.

Sets $F_n$ are closed, because norm is continuous and $L$ are continuous mappings. Because $X = \bigcup_n F_n$, there exists according to the Baire theorem such $k$ that the set $F_k$ has a nonempty interior. So exists $\epsilon > 0$ and $z \in F_k$ such that $\{x \in X : \|x - z\| < 2\epsilon\} \subset F_k$. Let’s fix $L \in G$ and $y \in X, \|y\| \leq 1$. Because $x \equiv z + \epsilon y \in F_k$, we will get $\|Ly\| = \left\|L\left(\frac{x - z}{\epsilon}\right)\right\| \leq \frac{1}{\epsilon}(\|Lx\| + \|Lz\|) \leq \frac{2k}{\epsilon}$, and therefore $\|L\| \leq \frac{2k}{\epsilon}$. □

The next result is the Banach - Steinhaus theorem:

**Theorem 1.2.** Let $X$ is a Banach space and $E$ is a normed linear space. Let’s $L_n \in L(X, E)$ is such a sequence, that exists $\lim L_n x$ for every $n$. We will denote such a limit $Lx$. Then $L \in L(X, E)$.

**Proof.** Direct application of principle of uniform boundedness. □

Next lemma is also a direct application of principle of uniform boundedness:

**Lemma 1.3.** Let be $\{x_n\}$ weakly convergent subsequence in normed linear space $E$. Then the sequence $\|x_n\|$ is strongly bounded.

**Proof.** We will choose $\varphi \in E^*$, then $\varphi(x_n)$ is convergent. So, there exists a constant $c_\varphi$, which depends only on $\varphi$, such that $|\varphi(x_n)| \leq c_\varphi$ for every $n$. We now define a mapping for every $n \in \mathbb{N}$

$$g_n : \varphi \mapsto \varphi(x_n) \text{ for } \varphi \in E^*,$$

we have $g_n \in E^{**}$ and $\|g_n\| = \|x_n\|$. Because the space $E^*$ is complete and the sequence $\{g_n\}$ is pointlike bounded, then the sequence of norms $\{|g_n|\}$ is according to principle of uniform boundedness bounded. □

**Lemma 1.4.** We can choose a weakly convergent subsequence from every bounded sequence in reflexive Banach space $X$.

**Proof.** Let $x_n$ is a bounded sequence in a reflexive Banach space $X$. We can consider a closed linear span of the sequence $x_n$, which we will denote $Y$. Then $Y$ is clearly separable and $Y^*$ is also separable. Let $\varphi_n$ is its dense subset. Because $\varphi_1(x_n)$ is bounded, we can choose a subsequence of $x_n, x_n^1$, such that $\varphi_1(x_n^1)$ is convergent. And we can further continue, we will choose a subsequence of $x_n^1$ such that $\varphi_2(x_n^2)$ is convergent, etc. Then finally $\varphi_k(x_n^k)$ converges for each $k$. Then it is not difficult
to show that $\varphi(x_n^n)$ must converge for every $\varphi \in Y^*$. If we choose $\varphi \in Y^*$ arbitrarily and $\epsilon > 0$, then we can choose $k$ such that $\|\varphi - \varphi_k\| < \epsilon$. If we will also denote $K = \sup\{\|x_n\| : n \in \mathbb{N}\}$, then it is according to the assumption $K < \infty$ and

$$|\varphi(x^i_n) - \varphi(x^j_n)| \leq 2K \|\varphi - \varphi_k\| + |\varphi_k(x^i_n) - \varphi_k(x^j_n)| \leq (2K + 1)\epsilon$$

for big enough $i, j$. Because $\varphi(x_n^n)$ is Cauchy, it is convergent. So we can define for $\varphi \in Y^*$

$$F(\varphi) = \lim n_x^n \varphi(x_n^n).$$

Because $\{x_n\}$ was bounded sequence, there is $F \in Y^{**}$. Because the space $Y$ is reflexive, there exists $x \in Y$ such that $\varphi(x) = F(\varphi)$ for every $\varphi \in Y^*$. Then $x^n \xrightarrow{w} x$.

Every continuous linear operator take weakly convergent sequences to weakly convergent sequences. Compact operator takes weakly convergent sequences to strongly convergent:

**Theorem 1.5.** Let be $M$ and $N$ normed linear spaces and $T$ is a compact operator. If $x_n \to x$, then $Tx_n \to Tx$.

**Proof.** Let’s suppose that $x_n \xrightarrow{w} 0$, then we have $T(x_n) \xrightarrow{w} T(0) = 0$. Let’s suppose that sequence $\{T(x_n)\}$ don’t converge to zero. Then there exists $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ such that $\|T(x_{n_k})\| \geq \epsilon$ for all $n_k$. According to one of the previous theorems is the whole sequence $\{x_n\}$ bounded. So bounded is also its subsequence $\{x_{n_k}\}$. From the definition of compactness of $T$ exists $y \in N$ and a subsequence $\{y_j\}$ from $\{x_{n_k}\}$, such that $Ty_j \xrightarrow{w} y$. But because $Ty_j \xrightarrow{w} 0$, then $y = 0$. Therefore $\|Ty_j\| \to 0$ and we get contradiction. □

**Definition 1.6.** We will say that a mapping $T \in L(X,Y)$ is open, if $T(G)$ is open subset of the range $T(X)$, whenever $G$ is open subset of $X$.

**Theorem 1.7.** Let be $X$ and $Y$ Banach spaces and $T \in L(X,Y)$. If the range of $T$ is $Y$, then $T$ is an open mapping.

**Proof.** We will write only the main idea of the proof. It is sufficient to prove that $TV$ is an open neighborhood of $0$, when $V$ is an open sphere with the center in zero. If $U$ is an open set and $y \in TU$, then we find $x \in U$ and $\epsilon > 0$ such that $Tx = y$ and $x + U(0,\epsilon) \subset U$. Then exists $\delta > 0$ such that $U(0,\delta) \subset TU(0,\epsilon)$. But then

$$y + U(0,\delta) \subset Tx + TU(0,\epsilon) = T(x + U(0,\epsilon)) \subset TU.$$ □

**Theorem 1.8.** If $X$ and $Y$ are Banach spaces and $T$ is an injective linear mapping $X$ onto $Y$ (surjective). Then $T^{-1}$ is continuous.

**Proof.** The proof follows from open mapping theorem. □

**Definition 1.9.** We say that a mapping $T : M \to N$, $M$ and $N$ are normed linear spaces, is closed, when the set $(x,Tx)$ is closed in $M \times N$. We could say alternatively that $T$ is closed when $z_n \to z$ and $T(z_n) \to y$

implies $Tz = y$.
Every continuous mapping is closed, but every closed mapping hasn’t to be continuous.

**Theorem 1.10.** Let be \( T : X \to Y \), closed mapping from Banach space \( X \) to Banach space \( Y \). Then it is continuous.

**Proof.** Graph of \( T \) is a closed subspace of Banach space \( X \times Y \). We define the mapping \( P : (x, Tx) \mapsto x \). Then the following holds:

\[
\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Lx\| = \|(x, Lx)\|
\]

Mapping \( P \) is linear, injective and surjective. \( \|P\| \leq 1 \). The inverse mapping \( P^{-1} \) is bounded according to closed mapping theorem. Then

\[
\|Lx\| \leq \|x\| + \|Lx\| = \|(x, Lx)\| = \|P^{-1}(x)\| \leq \|P^{-1}\| \|x\|.
\]

\[\square\]

It follows from this theorem:

**Lemma 1.11.** Let a Banach space is an algebraical sum of closed subspaces \( M \) and \( N \), \( X = M + N \). Then \( X \) is also their topological sum.