1. Weak and strong convergence

Topology defines the notion of convergence. When we change the topology, we change, which sequences will converge. This is the key concept. We should define, what is a topology. It is a subset $\tau \subset 2^X$ of the set of all subsets of a space X, such that the following three conditions are satisfied:

- $\emptyset, X \in \tau$
- $X_i \in \tau, (i \in \gamma$ could be uncountable$) \Longrightarrow \cup_{i=1}^{\infty} X_i \in \tau$
- $X_i \in \tau, (i = 1, ..., N; N \in \mathbb{N}) \Longrightarrow \cap_{i=1}^{N} X_i \in \tau$

Let's define first the convergence in Banach space X and its dual X^*. We say that a sequence $\{x_n\}$ convergences strongly, $x_n \to x$, if $\lim \|x_n - x\| = 0$.

When we consider the space of continuous linear forms on dual X^*, we define the norm standardly:

$$\|L\| = \sup\{|Lx| : \|x\|_X \leq 1\}, \ L \in X^* \tag{1}$$

This definition determines strong convergence on the dual.

When we have defined the notion of convergence in norm, we can define the sum of a serie, which is defined like a limit of sequence of partial sums followingly:

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k \tag{2}$$

We will now define two notions of weak convergence:

Definition 1.1. We say that $x_n \xrightarrow{w} x$ in X, if $f(x_n) \to f(x)$ for all $f \in X^*$.

Every strongly convergent sequence is also weakly convergent, but not contrary. It has also one limit, as can be seen from Hahn-Banach theorem.

Definition 1.2. We say that $L_n \xrightarrow{w^*} L$ in X^*, if $L_n(x) \to L(x)$ for all $x \in X$.

We can again see that every convergent sequence is also w^*-convergent, but not contrary. We can prove without Hahn-Banach theorem that it has one limit.

When X^* is a dual of Banach space X. Then we have two notions of weak convergencies on this dual. One is $L_n \xrightarrow{w} L$, if $\Phi(L_n) \to \Phi(L)$ for every $\Phi \in X^{**}$.The second one is $L_n \xrightarrow{w^*} L$, if $L_n(x) \to L(x)$ for all $x \in X$.

Lemma 1.3. *If X is reflexive*[1]*, then w- and w^*- convergence coincide.*

Proof. Follows directly from the definition of the reflexive space. □

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[1]We define two mappings ϵ and ϵ_x on X (this is a normed space) and X^*. ϵ is defined from X to X^{**} by the following prescription:

$$\epsilon : x \mapsto \epsilon_x,$$
$$\epsilon_x(\varphi) = \varphi(x),$$

where $\varphi \in X^*$.

ϵ is injective, isomorphic and isometric mapping on ϵX. ϵ is called a canonical embedding and it is a continuous linear form in X^*. We say that normed linear space is reflexive, if $\epsilon X = X^{**}$, where ϵ is the canonical embbeding.

We emphasize that a Banach space could be isometrically-isomorphic with its second dual and hasn't to be reflexive, because we talk about special mapping ϵ. Every Hilbert space is an example of reflexive space.

w-convergence generally doesn't need to coincide with w^*-convergence.

Definition 1.4. We say that a set C in metric space X is compact, if we can choose a finite covering from arbitrary open covering of this space. We could equivalently say that a set C is compact, if from arbitrary sequence we can choose a convergent subsequence with limit in such space.

Every compact set is closed and bounded. In finite dimensional spaces holds also that every closed and bounded set is compact. But in infinite dimensional spaces is every closed unit sphere never compact.

We say that a set M is totally bounded or precompact, if for every $\epsilon > 0$, we can choose a finite collection of points $x_1, x_2, ..., x_n$ such that $M \subset \cup_{i=1}^n U(x_i, \epsilon)$, where $U(x_i, \epsilon)$ are open balls with center in x_i.

We can prove that a set is precompact, if we could choose from every sequence Cauchy subsequence. Every precompact set is bounded. And compact set is every precompact set, which is complete.

A set is relatively compact, if its closure is compact. In complete metric spaces the notions precompact and relatively compact coincide. We say that a set is relatively compact if we can choose from every sequence a convergent subsequence.

Definition 1.5. We will say that the following set G, $G \subset X^*$ is w^*-open in Banach space X, if for all $\varphi \in G$ exist points $x_1, x_2, ..., x_n \in X$ and $\epsilon > 0$ such that

$$\{f \in X^* : \max\{|(f-\varphi)|(x_1), |(f-\varphi)|(x_2), ..., |(f-\varphi)|(x_n)\} < \epsilon\} \subset G \tag{3}$$

The system of w^*-open sets creates a topology on X^*. We call it w^* -topology.

Lemma 1.6. *Every w^*-open set is also (strongly) open. A sequence of functionals $f_n \in X^*$ converges to $f \in X^*$ in w^*-topology, if $f_n(x) \to f(x)$ for all $x \in X$.*

Proof. Immediately from definition. □

So, we see that this w^*-topology exactly prescribes the w^*-convergence. The w^*-topology coincides with normed topology in finite dimensional spaces. But on infinite dimensional spaces this two topologies never coincide. The w^*-topology is always Hausdorff. The following Alaoglu theorem is fundamental:

Theorem 1.7. *Let's X is a Banach space. Then closed unit sphere in X^* is always w^*-compact.*

Proof. If $\varphi \in B_{X^*}$ and $x \in X$, then $\varphi(x) \in [-\|x\|, \|x\|]$. If we consider mapping $\Theta : \varphi \mapsto \{\varphi(x)\}_{x\in X}$, Θ maps sphere B_{X^*} into cartesian product $\prod_{x\in X}[-\|x\|, \|x\|]$. The mapping Θ is injective.

Because w^*-topology on X^* is a topology of pointwise convergence on X, the mapping Θ is continuous, when we consider on X^* w^*-topology and on $\prod_{x\in x}[-\|x\|, \|x\|]$ topology of cartesian product. Therefore we can identify in this sense sphere B_{X^*} with some subset of $\prod_{x\in X}[-\|x\|, \|x\|]$.

According to Tichonoff theorem is $\prod_{x\in X}[-\|x\|, \|x\|]$ a compact space. And we need to prove that B_{X^*} is its w^*-closed subset.

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References

[1] J.Lukes, Zapisky z funcionalni analyzy
[2] Havlicek, Linearni operatory v kvantove fyzice