Plabic graphs in physics

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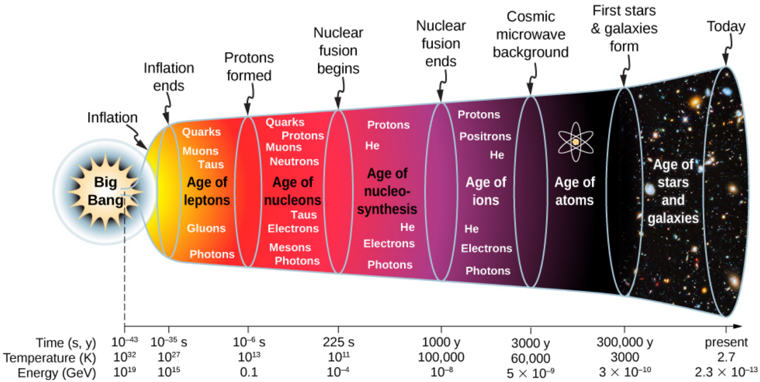
Seminar at Charles University in Prague, Department of applied mathematics, 7.5.2019

Accelerated expansion of the Universe

The Universe is expanding in the last 5 billion years.

Hot Big Bang Scenario



Quantum mechanics

- probability desctiption
- wave function
- Hilbert space
- self-adjoint operators

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

General theory of relativity

- geometrical theory
- spacetime
- metric
- Riemann tensor

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Non-locality

"In quantum mechanics quantum non-locality refers to the apparent instantaneous propagation of correlations between entangled systems, irrespective of their spatial separation. In quantum field theory, the notion of locality may have a different meaning."

Locality in QFT:

Algebras associated to spacelike separated regions commute: if O_1 is spacelike separated from O_2, then $[A, B] = 0, \forall A \in A(O_1), B \in A(O_2)$; This expresses the independence of physical systems associated to regions O_1 and O_2.

Issues in quantum gravity

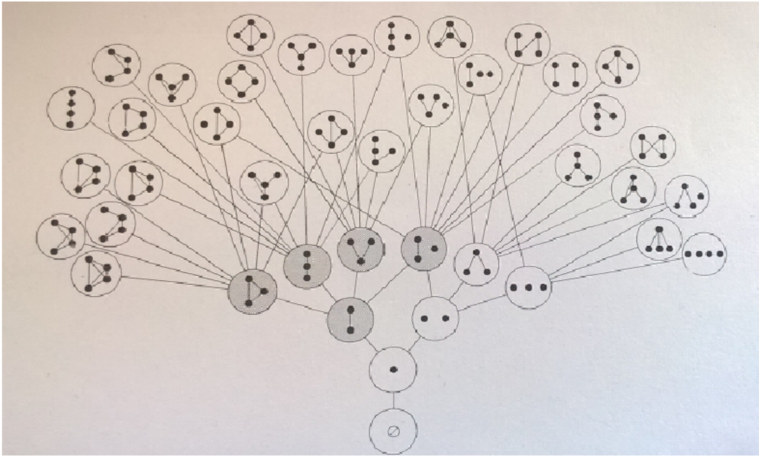
- non-locality
- background independence
- dimensional reduction

Axioms of irreflexive formulation of CST

The irreflexive formulation of causal set theory is defined by the following six axioms:

1. Binary axiom
2. Measure axiom
3. Countability
4. Transitivity
5. Interval finitness
6. Irreflexivity

Hasse



Particle in CST

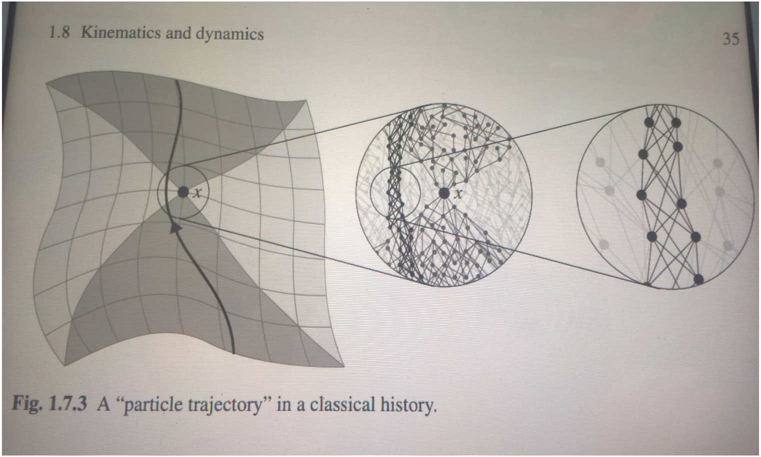

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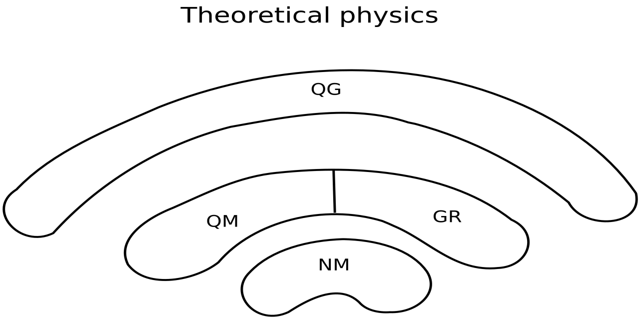
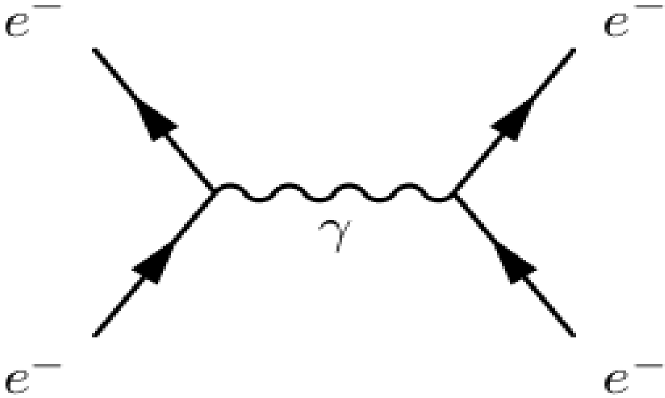
Fig. 1.7.3 A "particle trajectory" in a classical history.

Deductive layers of physics



Feynman diagram



Finite dimensional Feynman diagrams

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2}\, dx = \sqrt{\frac{2\pi}{a}}$$

More generally, let $A = A_{ij}$ be a real $d \times d$ **positive-definite matrix**, $x = (x_1, ..., x_d)$ the Euclidean coordinates in $V = \mathbb{R}^d$, and $\langle , \rangle : (\mathbb{R}^d)^* \times \mathbb{R}^d \to \mathbb{R}$ the standard pairing $\langle x_i, x_j \rangle = \delta^i_j$. Then

$$Z_0 = \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Ax, x\rangle} = (det\frac{A}{2\pi})^{-\frac{1}{2}}. \qquad (1)$$

Correlation functions

The **correlators** $\langle f_1, ..., f_m \rangle$ of m functions $f_i : \mathbb{R}^d \to \mathbb{R}$ are defined by plugging the product of these functions in the integrand and normalizing:

$$\langle f_1, f_2, ..., f_m \rangle = \frac{1}{Z_0} \int e^{-\frac{1}{2}\langle Ax, x\rangle} f_1(x) f_2(x) ... f_m(x)\, dx \qquad (2)$$

They may be computed using Z_b. Notice that

$$\frac{\partial}{\partial b_i} \int e^{-\frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle} = \int e^{-\frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle} x^i dx. \qquad (3)$$

Wick

$$\langle x^{i_1}, ...x^{i_m} \rangle = \partial_{i_1} ... \partial_{i_m} e^{\frac{1}{2}\langle b, A^{-1} b\rangle}|_{b=0}$$

Let $f_1(x), ..., f_m(x)$ be arbitrary linear functions of the coordinates x_i. Then all m**-point functions** vanish for odd m. For $m = 2n$ one has

$$\langle f_1, ..., f_m \rangle = \sum \langle f_{i_1}, f_{i_2} \rangle ... \langle f_{i_{m-1}}, f_{i_m} \rangle \tag{4}$$

Feynman diagrams

It is convenient to represent each term

$$\langle f_{i_1}, f_{i_2} \rangle ... \langle f_{i_{m-1}}, f_{i_m} \rangle \tag{5}$$

in **Wick's formula** by a simple graph. Consider m points, with the k-th point representing f_k. A pairing of $1, ..., 2n$ gives a natural way to connect these points by n edges, with an edge $e = (j, k)$ representing $A_e^{-1} = \langle f_j, A^{-1} f_k \rangle$. Equation becomes then

$$\langle f_1, ..., f_m \rangle = \sum_{\Gamma} \prod_{e \in edges\{\Gamma\}} A_e^{-1},$$

where the sum is over all univalent graphs as above.

Infinite-dimensional case

Instead of the **discrete set** $i \in \{1, ..., d\}$ of indices we have now **continuous variable** $x \in M^n$. So, the sum over i becomes an integral over x. Vectors $(x^1, x^2, ..., x^d)$ and $b = b(i)$ become fields $\phi = \phi(x)$ and $J(x)$. A quadratic form $A = A(i, j)$ becomes an integral kernel $K = K(x, y)$. Pairings $\langle Ax, x\rangle = \sum_{i,j} x^i A_{ij} x^j$ and $\langle b, x\rangle = \sum_i b^i x^i$ become $\langle K\phi, \phi\rangle = \int \phi(x) K(x, y) \phi(y)\, dxdy$ and $\langle J, \phi\rangle = \int J(x)\phi(x) dx$ respectively.

Amplituhedron

We start with **triangle in two dimensions.** When we think projectively, the vertices are Z_1^I, Z_2^I, Z_3^I, where $I = 1, 2, 3$. The interior of the triangle is a collection of points of the form

$$Y^I = c_1 Z_1^I + c_2 Z_2^I + c_3 Z_3^I \quad (6)$$

where we span over all c_a with $c_a > 0$. The interior of a triangle is associated with a triplet $(c_1, c_2, c_3)/GL(1)$ with all ratios $c_a/c_b > 0$, so that all c_a are either all positive or all negative.

One obvious generalization of the triangle is to an $(n-1)$ dimensional **simplex** in a general projective space, a collection $(c_1, ..., c_n)/GL(1)$, with $c_a > 0$. The n-tuple $(c_1, ..., c_n)/GL(1)$ specifies a line in n-dimensions, or a point in $\mathbb{P}^{n-1}$. We could generalize this to the space of k-planes in n dimensions - the Grassmannian $G(k, n)$, which we can take to be a collection of n k-dimensional vectors modulo $GL(k)$ transformations, grouped into a $k \times n$ matrix

$$C = (c_1 ... c_n)/GL(k).$$

Generalization of a triangle

One natural generalization of a triangle is to a more **general polygon** with n vertices $Z_1^I, ..., Z_n^I$. Once again we would like to discuss the interior of this region:

$$Y^I = c_1 Z_1^I + c_2 Z_2^I + ... + c_n Z_n^I, \; c_a > 0 \tag{7}$$

$$(c_1, ..., c_n) \subset G_+(1, n), \; Y^I = c_a Z_a^I, \; Z_1, ..., Z_n \subset M_+(1, n) \tag{8}$$

Generalization to higher projective spaces

This object has a natural generalization to **higher projective spaces**; we can consider n points Z_a^I in $G(1, 1+m)$, with $I = 1, ..., 1+m$, which are positive $\langle Z_{a_1} ... Z_{a_{1+m}} \rangle > 0$; Then, the analog of the inside of the polygon are points of the form $Y_I = c_a Z_a^I$, with $c_a > 0$.

Generalized amplituhedron

We can further generalize this structure into the **Grassmannian.** We take positive external data as $(k+m)$ dimensional vectors Z_a^I for $I = 1, ..., k+m$. It is natural to restrict $n \geq (k+m)$, so that the external Z_a fill out the entire $(k+m)$ dimensional space. Consider the space of k-planes in this $(k+m)$ - dimensional space, $Y \subset G(k, k+m)$, with co-ordinates $Y_a^I,\ a = 1, ...k,\ I = 1, ..., k+m.$

We then consider a subspace of $G(k, k+m)$ determined by taking all possible positive linear combinations of the external data, $Y = C.Z$ or more explicitly

$$Y_\alpha^I = C_{\alpha a} Z_a^I,$$

where
$C_{\alpha a} \subset G_+(k, n)$, $Z_a^I \subset M_+(k+m, n)$. It is trivial to see that this space is cyclically invariant if m is even: under the twisted cyclic symmetry, $Z_n \to (-1)^{k+m-1} Z_1$ and $c_n \to (-1)^{k-1} c_1$, and the product is invariant for even m. We call this space **the generalized tree amplituhedron** $A_{n,k,m}(Z)$.

Amplituhedron

Let Z be a $(k+m) \times n$ real matrix whose maximal minors are all positive, where $m \geq 0$ is fixed with $k+m \leq n$. Then it induces a map

$$\tilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$$

defined by $\tilde{Z}(\langle v_1, ..., v_k \rangle) \equiv \langle Z(v_1), ..., Z(v_k) \rangle$, where $\langle v_1, ..., v_k \rangle$ is an element of $Gr_{k,n}^{\geq 0}$ written as the span of k basis vectors. The (tree) **amplituhedron** $A_{n,k,m}(Z)$ is defined to be the image of $\tilde{Z}(Gr_{k,n}^{\geq 0})$ inside $Gr_{k,k+m}$.

Decorated permutation and Le-diagram

Definition

A decorated permutation of the set $[m]$ is a bijection $\pi : [m] \to [m]$ whose fixed points are colored either black or white. We denote a black fixed point $\pi(i) = \underline{i}$ and white fixed point by $\pi(i) = \bar{i}$. An antiexcendance of the decorated permutation π is an element $i \in [m]$ such that either $\pi^{-1}(i) > i$ or $\pi(i) = \bar{i}$ (i is a white fixed point).

Definition

Fix l and m. Given a partition λ, we let Y_λ denote the Young diagram associated to λ. A Le-diagram D of shape λ and type (l, m) is a Young diagram of shape Y_λ contained in a $l \times (m - l)$ rectangle, whose boxes are filled with 0 and 1 in such a way that the *Le*-property is satisfied: there is no 0 which has 1 above it in the same column and a 1 to its left in the same row.

We could obtain a bijection between *Le*-diagrams D of type (l, m) and decorated permutations π on $[m]$ with exactly l anti-excendances.

Algorithm

- Replace each 0 in the Le-diagram L with an elbow joint and each 1 in L with a cross $+$.
- The southeast border of Y_λ gives rise to a length-m path from the northeast corner to the southwest corner of the $l \times (m-l)$ rectangle. Label the edges of this path with the numbers 1 through m.
- Now label the edges of the north and west border of Y_λ so that opposite horizontal edges and opposite vertical edges have the same label.
- View the resulting 'pipe dream' as a permutation $\pi = \pi(L)$ on $[m]$, by following the 'pipes' from the southeaster border to the northwest border of the Young diagram. If the pipe originating at label j ends at label i, we define $\pi(j) = i$.
- If $\pi(j) = j$ and j labels two horizontal (respectively, vertical) edges of Y_λ, then $\pi(j) \equiv \underline{j}$ (respectively, $\pi(j) \equiv \bar{j}$).

Definition

A plabic graph is an undirected planar graph G drawn inside a disk (considered modulo homotopy) with m boundary vertices on the boundary of the disk, labeled $1, \ldots, m$ in clockwise order, as well as some colored internal vertices. These internal vertices are strictly inside the disk and are each colored either black or white. Each boundary vertex i in G is incident to a single edge. If a boundary vertex is adjacent to a leaf (vertex of degree 1), we refer to that leaf as a lollipop.

Definition

A **perfect orientation** P of a plabic graph H is a choice of orientation of each of its edges such that each black internal vertex v is incident to exactly one edge directed away from v, and each white internal vertex w is incident to exactly one edge directed towards w. A plabic graph is called **perfectly orientable** if it admits a perfect orientation. Let H_o denote the directed graph associated with a perfect orientation P of H. Since each boundary vertex is incident to a single edge is either source (if it is incident to an outgoing edge) or a sink (if it is incident to an incoming edge) in H_o. The source set $J_0 \subset [m]$ is the set of boundary vertices, which are sources in H_0.

Plabic graph and Le-diagram

The following construction associates a perfectly orientable **plabic graph** to any **Le-diagram.**

Let L be a Le-diagram and σ its decorated permutation. Delete the 0's of D, and replace it with a vertex. From each vertex we construct a **hook** which goes east and south, to the border of the Young diagram. The resulting diagram is called a **hook diagram** $H(L)$. After replacing the edges along the southeast border of the Young diagram with boundary vertices labeled by $1, ..., m$, we obtain a planar graph in a disk, with m boundary vertices and one internal vertex for each $+$ of L. Then we replace the local region around each internal vertex as in Figure, and add a black **lollipop** for each black fixed point of σ. This gives rise to a plabic graph which we call $H(L)$. By orienting the edges of $H(L)$ down and to the left, we obtain a perfect orientation.

Grassmannians

Definition

The (real) **Grassmannian** Gr_{lm} is the space of all l-dimensional subspaces of $\mathbb{R}^m$, for $0 \leq p \leq m$. An element of Gr_{lm} can be viewed as $l \times m$ matrix of rank l, modulo left multiplications by non-singular $l \times l$ matrices.
Let $[m]$ denote $\{1, ..., m\}$, and $\binom{[m]}{l}$ the set of all l-element subsets of $[m]$. Given $W \in Gr_{lm}$ represented by $l \times m$ matrix A, for $J \in \binom{[m]}{l}$ we let $\Delta_J(W)$ be the maximal minor of A located in the column set J. The $\Delta_J(W)$ do not depend on our choice of matrix A and are called the **Plücker coordinates** of W.

Definition

The **totally nonnegative Grassmanian** $Gr_{kn}^{\geq 0}$ is the set of elements $V \in Gr_{kn}$ such that $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$. For $M \subseteq \binom{[n]}{k}$, the **positive Grassmann cel** S_M is the subset of elements $V \in Gr_{kn}^{\geq}$ with the prescribed collection of Plücker coordinates strictly positive and the remaining Plücker coordinates equal to zero. We call M a positroid if S_M is nonempty.

Definition

A **planar directed graph** G is a directed graph drawn inside a disk. We will assume that G has finitely many vertices and edges. We allow G to have loops and multiple edges. We will assume that G has n **boundary vertices** on the boundary of the disk labelled $b_1, ..., b_n$ clockwise. The remaining vertices, called the internal vertices, are located strictly inside the disk. We will always assume that each boundary vertex b_i is either a source or a sink. Even if b_i is an isolated boundary vertex, we will assign b_i to be source or a sink. A **planar directed network** $N = (G, x)$ is a planar directed graph G as above with strictly positive real weights $x_e > 0$ assigned to all edges e of G.

Definition

For such network N, the **source set** $I \subset [n]$ and the sink set $\bar{I} \equiv [n] \setminus I$ of N are the sets such that $b_i, i \in I$, are the sources of N and the $b_j, j \in \bar{I}$, are the boundary sinks.

Definition

If the network N is acyclic, that is it does not have closed directed paths, then, for any $i \in I$ and $j \in \bar{I}$, we define the boundary measurement M_{ij} as the finite sum

$$M_{ij} \equiv \sum_{P: b_i \to b_j} \prod_{e \in P} x_e,$$

where the sum is over all directed paths P in N from the boundary source b_i to the boundary sink b_j, and the product is over all edges e in P.

Winding index

For a path P from a boundary vertex b_i to a boundary vertex b_j, we define its **winding index**, as follows.

Definition

We can now define the winding index $wind(P) \in \mathbb{Z}$ of the path P as the signed number of full 360° turns the tangent vector $f'(t)$ makes as we go from b_i to b_j (counting counterclockwise turns as positive). Similarly, we define the winding index $wind(C)$ for a closed directed path C in the graph.

Recursive combinatorical procedure

Let us give a recursive combinatorical procedure for calculation of the winding index for a path P with vertices $v_1, v_2, ..., v_l$. If the path P has no self-intersections, $wind(P) = 0$. Also for a counterclockwise (clockwise) closed path C without self-intersections, we have $wind(C) = 1$ ($wind(C) = -1$).

Suppose that P has at least one self-intersection, that is $v_i = v_j = v$ for $i < j$. Let C be the closed segment of P with $v_i, v_{i+1}, ..., v_j$ and let P' be the path with erased segment C, P' has the vertices $v_1, ..., v_i, v_{j+1}, ..., v_l$.

Consider the four edges $e_1 = (v_{i-1}, v_i), e_2 = (v_i, v_{i+1}), e_3 = (v_{j-1}, v_j), e_4 = (v_j, v_{j+1})$ in the path P, which are incident to the vertex v (the edges e_1 and e_3 are incoming , and the edges e_2 and e_4 are outgoing). Define the number $\epsilon = \epsilon(e_1, e_2, e_3, e_4) \in \{-1, 0, 1\}$, as follows. If the edges are arranged as e_1, e_2, e_3, e_4 clockwise, then set $\epsilon = -1$; otherwise set $\epsilon = 0$. In particular, if some of the edges e_1, e_2, e_3, e_4 are the same, then $\epsilon = 0$. Informally, $\epsilon = \pm 1$, if the path P does not cross but rather touches itself at the vertex v.

Lemma

We have $wind(P) = wind(P') + wind(C) + \epsilon$.

Let N be a planar directed network with graph G as above, which is now allowed to have cycles. Let us assume for a moment that the weights x_e of edges in N are formal variables. For a path P in G with the edges $e_1, ..., e_l$, we will write $x_P = x_{e_1} ... x_{e_l}$. For a source $b_i, i \in I$, and a sink $b_j, j \in \bar{I}$, we define the formal boundary measurement M_{ij}^{form} as the formal series in the x_e

$$M_{ij}^{form} \equiv \sum_{P: b_i \to b_j} (-1)^{wind(P)} x_P,$$

where the sum is over all directed paths P in N from b_i to b_j.

Recall that a **subtraction-free rational expression is an expression** with positive integer coefficients that can be written with the operations of addition, multiplication, and division.

$$\frac{x + y/x}{z^2 + 25y/(x+s)} = \frac{(x^2+y)(x+s)}{z^2x(x+s) + 25xy}$$

- We can now define the **boundary measurements** M_{ij} as the specializations of the **formal boundary measurements** M_{ij}^{form}, written as subtraction-free expressions, when we assign the x_e to be the positive real weights of edges e in the network N
- What information about a planar directed network can be recovered from the collection of boundary measurements M_{ij}? How to recover this information?

Let Net_{kn} be the set of planar directed networks with k boundary sources and $n - k$ boundary sinks. Define the **boundary measurement map**

$$Meas : Net_{kn} \to Gr_{kn} \tag{9}$$

followingly. For a network $N \in Net_{kn}$ with the source set I and with the boundary measurements M_{ij}, the point $Meas(N) \in Gr_{kn}$ is given in terms of its Plücker coordinates by the conditions that $\Delta_I \neq 0$ and

$$M_{ij} = \Delta_{(I \setminus \{i\}) \cup \{j\}} / \Delta_I, \tag{10}$$

for any $i \in I$ and $j \in \bar{I}$.

More, explicitely if $I = \{i_1 < i_2 < ... < i_k\}$, then the point $Meas(N) \in Gr_{kn}$ is represented by the boundary measurement matrix $A(N) = (a_{ij}) \in Mat_{kn}$ such that

1. The submatrix $A(N)_I$ in the column set I is the identity matrix Id_k.
2. The remaining entries of $A(N)$ are $a_{rj} = (-1)^s M_{i_r,j}$, for $r \in [k]$ and $j \in \bar{I}$, where s is the number of elements of I strictly between i_r and j.

Theorem

The image of the **boundary measurement map** *Meas is exactly the totally non-negative Grassmanian:*

$$Meas(Net_{kn}) = Gr_{kn}^{\geq 0}$$

Feynman diagrams and plabic graphs

One can think about plabic graphs as some kind of Feynman diagrams, where the black and white vertices represent certain elementary particles of two types and edges represent interactions between these particles.

- plabic graphs: inner structure in Feynman diagrams
- road to Quantum Gravity ?
- many open problems in this part of algebraic geometry

Pictures taken from web.

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