What is game theory

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Many people, who are not working in sicence, are asking a simple question, why is mathematics so important. I want to show on three simple examples, what is game theory (branch of mathematics) and for what is it good for.

Let's start with chess, Figure 1. There are some rules according to which we could move the chessmen. The two players are using different strategies to beat their opponent. The subject matter of game theory are exactly interactions within a group of individuals, where the actions of each individual have an effect on the outcome that is of interest to all. Game Theory (GT) studies strategic interactions between individuals.

The most important assumption of GT which started the research in this discipline is that individuals are rational. We start with this definition:

Definition 1. *An individual is rational if he has well-defined objectives (or preferences) over the set of possible outcomes and he implements the best available strategy to pursue them.*

Now we could define precisely what GT is:

Definition 2. *GT is a systematic study of strategic interactions among rational individuals.*

We will start with an example in which there is no strategic interaction. It is the single person decision problem:

Example 1. *We will suppose that Johny is an investor who can invest his* \$10 *to a safe asset (for example govermental bonds), which brings* 10% *return in one year, or he could invest it in a risky asset (for example stock issued by a corporation) which either brings* 20% *return back (if the company performance is good) or no return (if the company performance is bad).*

The investment, which is good for him depends on his preferences and the relative likelihoods of the states of the world. We will denote the probability of the good state p *and that of the bad state* $1-p$*. Assume that Johny wants to maximize the amount of the money he has at the end of the year. If he invests his* \$10 *on bonds, he will have* \$11 *with certainity at the end of the year irrespective of the state of the world. If he*

Figure 1: Many people like to play chess. Chess is an example of a game with two players.

invests on stocks, he will get \$12 *with probability* p *and* \$10 *with probability* $(1-p)$. *Therefore we could calculate his average money holdings at the end of the year:*

$$p.12 + (1-p).10 = 10 + 2.p$$

So, it is useful to invests into stocks, if the probability is $p > \frac{1}{2}$, *Table 1.*

The decision in this example is a problem of an individual and can be analyzed in isolation of the behavior of other individuals. The other two examples are different.

Table 1: Johny invests to bonds or stocks, first and second row in the table.

	Good	Bad
Bonds	10%	10%
Stocks	20%	0%

The second example is an investment game:

Example 2. *We suppose that Johny has again two options, how he will invest his* \$10. *He could invest in bonds, then he get* 10% *at the end of the year, or he could try a risky venture. The investement must be* \$20 *for success, in which case the return is* 20% , *and this is* \$12 *at the end of the year. If total investment is less than* \$20, *then the venture is a failure and we have no return, so* \$10 *investment yields* \$10. *But Johny knows that there is another person, Kate, who is exactly in the same situation. There is no other person. Unfortunately, Johny and Kate don't know each other and they cannot communicate. Therefore, they must both make the investment decision without knowing the decisions of each other.*

We can summarize the returns on the investments of Johny and Kate as a function of their decisions, Table 2. The first number in each column represents Johny's return and the second number Kate's return. Both Johny and Kate know the rules of the game.

The new phenomenon here is the existence of strategic interaction. The key element is that the outcome of Johny's decision depends on what Kate did. Investing in the risky option, venture, has an uncertain return, as it was the case in previous example. But now is the source of the uncertainty another individual, Kate. If Johny believes that Kate is going to invest in the venture, then his best choice is the venture as well. If he thinks Kate is going to invest in bonds, his optimal choice is to invest in bonds as well. Kate is in a similar situation.

Table 2: The second example is an investment game. Johny and Kate could choose two different strategies. They could invest to bonds or they could choose venture. We have the following diagram of returns. The left number is Johny's return and the right number Kate's return.

	Bonds	Venture
Bonds	11,11	11,10
Venture	10,11	12,12

Now is the question, what will Johny do? We don't have enough information to provide an answer at this moment. We want to describe Johny's and Kate's objectives, which are their preferences over the set of possible outcomes. One favorite possibility of economists, is to assume that they are both expected, that they will maximize their profit. But this is not enough for us to answer Johny's question. We have to give Johny a way to form expectations what he thinks that Kate will do.

The simplest possibility is to assume that Johny expects Kate is going to choose bonds with some probability p between zero and one. But then the decision problem becomes identical to the one in Example 1. It is not necessary to use game theory in this case. But how we know that Kate will decide in this mechanical way? We only know that Kate is also money maximizer. So, let's assume that they are both rational and they both know that the other is rational.

Should we add something ? Johny knows that Kate is rational. But this not enough for him to deduce what she will do. He knows that she will act in the way what would maximize her expected return. This depends on the way what she thinks Johny is going to do. Therefore, what Johny should do depends on what Kate thinks that he is going to do. We could continue in this way that both know that both know that both are rational. And game theory solves this kind of problems. The next example deals with a problem which is still easy to solve.

Example 3. *We will describe a game, which is called the Prisoners' Dilemma. The story is the following: two suspects are arrested and they put them into different rooms before the trial. The attorney, who is pretty sure that both of the suspects are guilty but lacks enough evidence, offers them the following choice: if both of them confess and they will confirm that the other (labeled C) is guilty, then they will be sentenced to, say, 5 years of prison time. If one confesses and the other does not (labeled N), then the saboteur goes free for his cooperation with the authorities and the non-confessor is sentenced to 6 years of prison time. Finally, if neither of them confesses, then both suspects must be in prison one year.*

We could represent this game as in Table 2 where we assume that the utility of a year in prison is -1 for each guy.

Table 3: The third example is prisoner's dilemma. Johny and Kate could choose from two different strategies. They could confess (C) or not confess (N). The length of their stay in prison will be according to their decision.

	C	N
C	-5,-5	-6, 0
N	0,-6	-1,-1

The best outcome for Johny is the case in which he confesses and Kate does not. The next best outcome for Johny is (N, N), and then (C, C) and finally (N, C). A similar interpretation applies to Kate. How we would look at this situation from the point of view of Johny? The important observation is the following: we don't care what Kate intends to do, playing C yields a better outcome for Johny. The reason for this is that (C, C) is a better outcome for Johny than (N, C), and (C, N) is a better outcome for him than (N, N). Therefore it seems only good choice for Johny to play C. The same reasoning for Kate causes that she is very likely to play C. Therefore the very reasonable prediction here is that the game will end in the outcome (C, C) when both players confess to their crimes.

And here comes the dilemma: isn't a better a choice for both players to play N instead? After all, (N, N) is a better choice by both players to (C, C). It is really their pity that the rational individualistic play leads to a worth outcome from the perspective of both guys. You could interpret it that this situation arises here only because the prisoners are in separate cells and they could not communicate. You might argue that if the players start to debate about how to play this game, they would find out that (N, N) is a better choice relative to (C, C) for both of them. Thus they would agree to play N instead of C. But even if such a pre-agreement is reached prior to the actual play of the game, what makes Johny so sure that Kate will not cheat him in the last instant by playing C, if Kate is sure that Johny will keep his end of the better choice by playing N, it is better for her to play C. Thus, even if such players reach pre-agreement, both of them may reasonably fear the betray. Therefore the better choice for them is to betray before being betrayed by playing C. And the dilemma is back.

These three examples nicely illustrate, what type of problems we study in game theory. As we saw, this discipline had natural applications to economics and politology. We will show connections to evolutionary biology. We again use an example.

Evolutionary stable strategies

We will consider particular species of beetle, and we suppose that each beetle's fitness in the given environment is determined largely by the extent to which it could find nutrition and use the nutrients from the food effectively. It is supposed that a particular mutation is introduced into the population and the beetles with the mutation grow signficantly larger. There are now two kinds of beetles in the population — small ones and large ones. The large beetles need to maintain the metabolic requirements of their larger body size and so this has a negative effect on their fitness. If this would be all, we could conclude that the large-body-size mutation will likely be driven out of the population over time, through multiple generations. But in fact, there's more to add, as we'll see, because there is interaction among organisms.

The beetles in this population compete with each other for food. When they find a food source, there's a battle among the beetles as they each try to get as much of the food as they could. And, not surprisingly, the beetles with large body sizes are more effective in the fight for the food. Let's assume for simplicity that food competition in this population involves two beetles. When two beetles compete for some food, we have the following possible outcomes:

1. When beetles of the same size compete, they share similar portions of the food.
2. When a large beetle competes with a small beetle, the large beetle gets more food.

In both cases large beetles get less of a fitness benefit from a given quantity of food, since some of it is converted to a source of nutrition for their organisms. The fitness that each beetle gets from a given interaction can be thought of as a numerical payoff in a two-player game between a first and a second beetle. We describe the situation again on the Table 4. The first beetle chooses one of the two strategies Small or Large, depending on the body size. The second beetle plays one of the two strategies as well.

Table 4:

	Small	Large
Small	5, 5	1, 8
Large	8, 1	3, 3

The numerical payoffs satisfy the principles just outlined: when two small beetles meet, they share the fitness from the food equally; large beetles do well at the presence of small beetles; but large beetles cannot get the full amount of fitness from the food source. The payoff matrix is a nice way to summarize what will happen when two beetles meet. But there is a new phenomenon. Each of the beetles are genetically hard-wired to play one of these two strategies through its whole lifetime. So the idea of choosing strategies is completely missing here. We will need to think about strategy changes that operate over longer time scales, taking place as shifts in a population under evolutionary forces.

Nash equilibrium is central in reasoning about the outcome of a game. It was studied by John Forbes Nash Jr. in last century, Figure 2. In a Nash equilibrium for a two-player game, neither player has an inducement to deviate from the strategy they are currently using. The equilibrium is a choice of strategies that tends to persist once the players start to use it. The analogous notion for evolutionary-type of games will be that of an evolutionarily stable strategy.

Figure 2: Mathematician Dr. John Forbes Nash, Jr. (1928 -2015)

We formulate it followingly. We consider again our example and we suppose that each beetle is repeatedly paired off with other beetle in food competitions over the course of its lifetime. The assumption is that the population is large enough that no two particular beetles have a signficant probability of interacting with each other repeatedly. The overall fitness of every beetle will be equal to the average fitness it experiences from each of its many pairwise interactions with others, and this overall fitness determines the reproductive success. This means that we need to count the number of offsprings that carry its genes into the next generation. We say that a given strategy is evolutionarily stable if the whole population is using this strategy and any small group of invaders using a different strategy will eventually die off over multiple generations. We describe this idea in terms of numerical payoffs by saying that when the whole population is using a strategy S, then a small group of invaders using any alternate strategy T should have strictly lower fitness than the users of the dominant strategy S. Since the fitness translates into reproductive success, evolutionary principles dictates that strictly lower fitness is the condition that causes a sub-population (like the users of strategy T) to shrink over time, through multiple generations. They will eventually die off with high probability. More formally, we will phrase the basic definitions as follows:

1. We say that the fitness of an organism in our population is the expected payoff that it receives from an interaction with a random member of the population.

2. We say that a strategy T invades a strategy S at level x, for some small positive

number x, if an x fraction of the underlying population uses T and a $1 - x$ fraction of the underlying population uses S.

3. We say that a strategy S is evolutionarily stable if there is a small positive number y such that when any other strategy T invades S at any level $x < y$, the fitness of an organism playing S is strictly greater than the fitness of an organism playing T.

Let's discuss what happens when we apply this definition to our example involving beetles, which are competing for food. First we will check whether the strategy Small is evolutionarily stable, and then we will do the same for the strategy Large. We will follow the definition and we suppose that for some small positive number x, $1 - x$ fraction of the population uses Small and x fraction of the population uses Large.

1. So, what is the expected payoff to a small beetle in a random interaction in this population? We see that with probability $1 - x$, it meets another small beetle, receiving a payoff of 5, while with probability x, it meets a large beetle, receiving a payoff of 1. Therefore its expected payoff is $5(1 - x) + 1x = 5 - 4x$.

2. What payoff we will expect to a large beetle in a random interaction in this population? With probability $1 - x$, it meets a small beetle, receiving a payoff of 8, while with probability x, it meets another large beetle, receiving a payoff of 3. Therefore its expected payoff is $8(1 - x) + 3.x = 8 - 5x$.

   We could easily check that for small enough values of x (and even for reasonably large ones in this case), the expected fitness of large beetles in this population exceeds the expected fitness of small beetles. Therefore Small is not evolutionarily stable.

References

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