Einstein meets Grothendieck: RT-paradigm as Quantum Gravity

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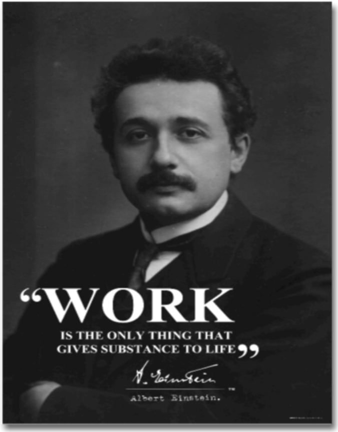
Albert Einstein



Alexander Grothendieck

Martin Doubek

Inspiration from philosophy

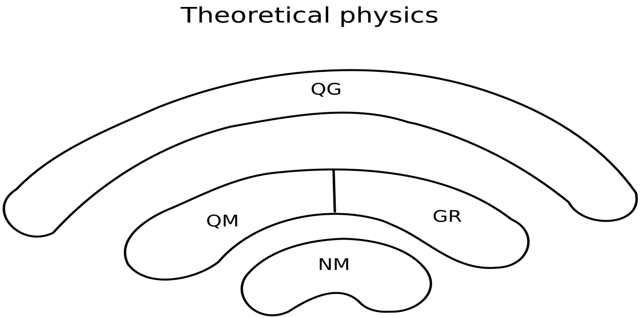
- finitism: avoids completed infinities altogether
- it seems that no question about experimental consequences on physical theories has an answer that depends on whether or not we assume the continuum hypothesis

RT-paradigm

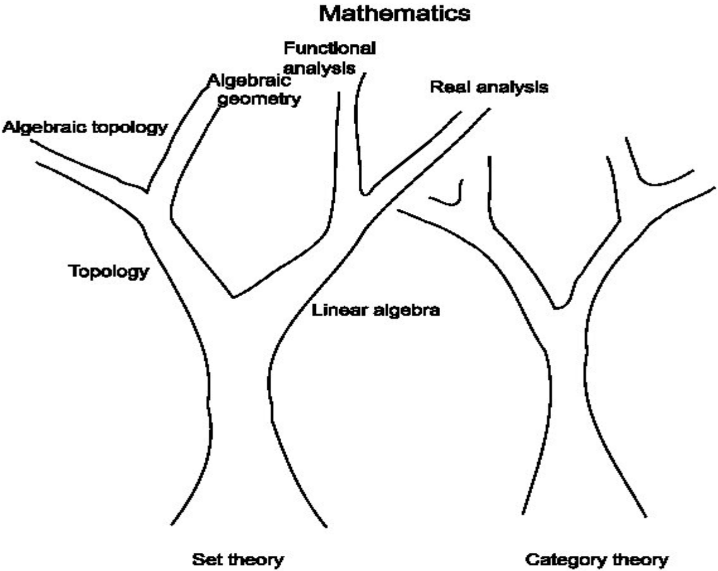
(ST)RING THEORY: $4 \to 3$

- RT-paradigm: collection of ideas, from which could be build RT (physical theory: Ring Theory)
- philosophy, physics, **mathematical apparatus**

Deductive layers of physics



Inductive trees of mathematics



Quantum Gravity

Common features for all approaches to QG: String theory (ST), Loop quantum gravity (LQG), Causal set approach (CSA), Causal dynamical triangulations (CDT), Regge calculus (RC) etc.

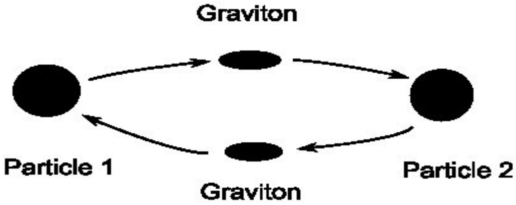
Common features

- Nonlocality
- Background independence, especially problem of time
- Dimensional reduction
- Determinism/Indeterminism
- Problem of singularities

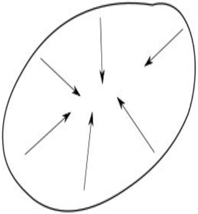
1. Dark energy
2. Arrow of time
3. Connection to particle physics
4. Dark matter
5. EPR experiment and Wheeler delayed choice experiment
   Please, see my poster:

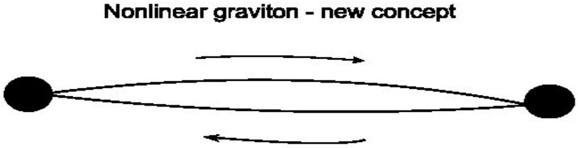
RT-paradigm as Quantum Gravity

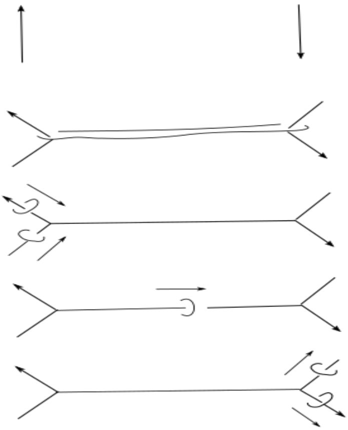
Old concept



Gravitating ring




Nonlinear graviton - new concept



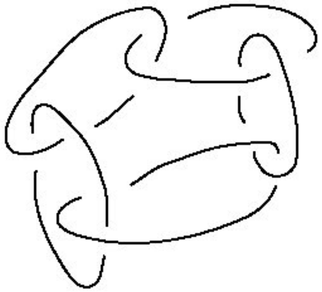
Mathematical apparatus of RT-paradigm

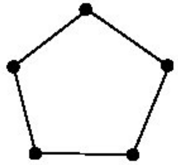
We will call, that a circle $S^1 \subset \mathbb{R}^3$ with finite length and finite circumference (we have a picture of torus in our mind), which could be deformed, is a **ring.**

Algebraic topology in RT-paradigm

Mathematical problem: we have a finite collection of N rings S^1 in $\mathbb{R}^3$, which could not touch; Give a complete characterization of all non-homeomorphic structures, which could be constructed from this finite collection of rings. Every two rings could be linked only once, they could not be knotted or twisted (in the case we have differentiable structure). We do not consider any Brunnian type of link in 3 and more rings.

Hopf-linked rings and plabic graph





Decorated permutation and Le-diagram

Definition

A decorated permutation of the set $[m]$ is a bijection $\pi : [m] \to [m]$ whose fixed points are colored either black or white. We denote a black fixed point $\pi(i) = \underline{i}$ and white fixed point by $\pi(i) = \bar{i}$. An antiexcendance of the decorated permutation π is an element $i \in [m]$ such that either $\pi^{-1}(i) > i$ or $\pi(i) = \bar{i}$ (i is a white fixed point).

Definition

Fix l and m. Given a partition λ, we let Y_λ denote the Young diagram associated to λ. A Le-diagram D of shape λ and type (l, m) is a Young diagram of shape Y_λ contained in a $l \times (m - l)$ rectangle, whose boxes are filled with 0 and 1 in such a way that the *Le*-property is satisfied: there is no 0 which has 1 above it in the same column and a 1 to its left in the same row.

We could obtain a bijection between *Le*-diagrams D of type (l, m) and decorated permutations π on $[m]$ with exactly l anti-excendances.

Algorithm

- Replace each 0 in the Le-diagram L with an elbow joint and each 1 in L with a cross $+$.
- The southeast border of Y_λ gives rise to a length-m path from the northeast corner to the southwest corner of the $l \times (m - l)$ rectangle. Label the edges of this path with the numbers 1 through m.
- Now label the edges of the north and west border of Y_λ so that opposite horizontal edges and opposite vertical edges have the same label.
- View the resulting 'pipe dream' as a permutation $\pi = \pi(L)$ on $[m]$, by following the 'pipes' from the southeaster border to the northwest border of the Young diagram. If the pipe originating at label j ends at label i, we define $\pi(j) = i$.
- If $\pi(j) = j$ and j labels two horizontal (respectively, vertical) edges of Y_λ, then $\pi(j) \equiv \underline{j}$ (respectively, $\pi(j) \equiv \bar{j}$).

Definition

A plabic graph is an undirected planar graph G drawn inside a disk (considered modulo homotopy) with m boundary vertices on the boundary of the disk, labeled $1, ..., m$ in clockwise order, as well as some colored internal vertices. These internal vertices are strictly inside the disk and are each colored either black or white. Each boundary vertex i in G is incident to a single edge. If a boundary vertex is adjacent to a leaf (vertex of degree 1), we refer to that leaf as a lollipop.

Definition

A **perfect orientation** P of a plabic graph H is a choice of orientation of each of its edges such that each black internal vertex v is incident to exactly one edge directed away from v, and each white internal vertex w is incident to exactly one edge directed towards w. A plabic graph is called **perfectly orientable** if it admits a perfect orientation. Let H_o denote the directed graph associated with a perfect orientation P of H. Since each boundary vertex is incident to a single edge is either source (if it is incident to an outgoing edge) or a sink (if it is incident to an incoming edge) in H_o. The source set $J_0 \subset [m]$ is the set of boundary vertices, which are sources in H_0.

Plabic graph and Le-diagram

The following construction associates a perfectly orientable **plabic graph** to any **Le-diagram.**

Let L be a Le-diagram and σ its decorated permutation. Delete the 0's of D, and replace it with a vertex. From each vertex we construct a **hook** which goes east and south, to the border of the Young diagram. The resulting diagram is called a **hook diagram** $H(L)$. After replacing the edges along the southeast border of the Young diagram with boundary vertices labeled by $1, ..., m$, we obtain a planar graph in a disk, with m boundary vertices and one internal vertex for each $+$ of L. Then we replace the local region around each internal vertex as in Figure, and add a black **lollipop** for each black fixed point of σ. This gives rise to a plabic graph which we call $H(L)$. By orienting the edges of $H(L)$ down and to the left, we obtain a perfect orientation.

Grassmannians

Definition

The (real) **Grassmannian** Gr_{lm} is the space of all l-dimensional subspaces of $\mathbb{R}^m$, for $0 \le p \le m$. An element of Gr_{lm} can be viewed as $l \times m$ matrix of rank l, modulo left multiplications by non-singular $l \times l$ matrices.
Let $[m]$ denote $\{1, ..., m\}$, and $\binom{[m]}{l}$ the set of all l-element subsets of $[m]$. Given $W \in Gr_{lm}$ represented by $l \times m$ matrix A, for $J \in \binom{[m]}{l}$ we let $\Delta_J(W)$ be the maximal minor of A located in the column set J. The $\Delta_J(W)$ do not depend on our choice of matrix A and are called the Plücker coordinates of W.

Definition

The **totally nonnegative Grassmanian** Gr_{kn}^{tnn} is the set of elements $V \in Gr_{kn}$ such that $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$. For $M \subseteq \binom{[n]}{k}$, the **positive Grassmann cel** S_M is the subset of elements $V \in Gr_{kn}^{tnn}$ with the prescribed collection of Plücker coordinates strictly positive and the remaining Plücker coordinates equal to zero. We call a M a positroid if S_M is nonempty.

Definition

A **planar directed graph** G is a directed graph drawn inside a disk. We will assume that G has finitely many vertices and edges. We allow G to have loops and multiple edges. We will assume that G has n **boundary vertices** on the boundary of the disk labelled $b_1, ..., b_n$ clockwise. The remaining vertices, called the internal vertices, are located strictly inside the disk. We will always assume that each boundary vertex b_i is either a source or a sink. Even if b_i is an isolated boundary vertex, we will assign b_i to be source or a sink. A **planar directed network** $N = (G, x)$ is a planar directed graph G as above with strictly positive real weights $x_e > 0$ assigned to all edges e of G.

Definition

For such network N, the **source set** $I \subset [n]$ and the sink set $\bar{I} \equiv [n] \setminus I$ of N are the sets such that $b_i, i \in I$, are the sources of N and the $b_j, j \in \bar{I}$, are the boundary sinks.

Definition

If the network N is acyclic, that is it does not have closed directed paths, then, for any $i \in I$ and $j \in \bar{I}$, we define the boundary measurement M_{ij} as the finite sum

$$M_{ij} \equiv \sum_{P: b_i \to b_j} \prod_{e \in P} x_e,$$

where the sum is over all directed paths P in N from the boundary source b_i to the boundary sink b_j, and the product is over all edges e in P.

Winding index

For a path P from a boundary vertex b_i to a boundary vertex b_j, we define its **winding index**, as follows.

Definition

We can now define the winding index $wind(P) \in \mathbb{Z}$ of the path P as the signed number of full 360° turns the tangent vector $f'(t)$ makes as we go from b_i to b_j (counting counterclockwise turns as positive). Similarly, we define the winding index $wind(C)$ for a closed directed path C in the graph.

Recursive combinatorical procedure

Let us give a recursive combinatorical procedure for calculation of the winding index for a path P with vertices $v_1, v_2, ..., v_l$. If the path P has no self-intersections, $wind(P) = 0$. Also for a counterclockwise (clockwise) closed path C without self-intersections, we have $wind(C) = 1$ ($wind(C) = -1$).

Suppose that P has at least one self-intersection, that is $v_i = v_j = v$ for $i < j$. Let C be the closed segment of P with $v_i, v_{i+1}, ..., v_j$ and let P' be the path with erased segment C, P' has the vertices $v_1, ..., v_i, v_{j+1}, ..., v_l$.

Consider the four edges $e_1 = (v_{i-1}, v_i), e_2 = (v_i, v_{i+1}), e_3 = (v_{j-1}, v_j), e_4 = (v_j, v_{j+1})$ in the path P, which are incident to the vertex v (the edges e_1 and e_3 are incoming , and the edges e_2 and e_4 are outgoing). Define the number $\epsilon = \epsilon(e_1, e_2, e_3, e_4) \in \{-1, 0, 1\}$, as follows. If the edges are arranged as e_1, e_2, e_3, e_4 clockwise, then set $\epsilon = -1$; otherwise set $\epsilon = 0$. In particular, if some of the edges e_1, e_2, e_3, e_4 are the same, then $\epsilon = 0$. Informally, $\epsilon = \pm 1$, if the path P does not cross but rather touches itself at the vertex v.

Lemma

We have $wind(P) = wind(P') + wind(C) + \epsilon$.

Let N be a planar directed network with graph G as above, which is now allowed to have cycles. Let us assume for a moment that the weights x_e of edges in N are formal variables. For a path P in G with the edges $e_1, ..., e_l$, we will write $x_P = x_{e_1} ... x_{e_l}$. For a source $b_i, i \in I$, and a sink $b_j, j \in \bar{I}$, we define the formal boundary measurement M_{ij}^{form} as the formal series in the x_e

$$M_{ij}^{form} \equiv \sum_{P: b_i \to b_j} (-1)^{wind(P)} x_P,$$

where the sum is over all directed paths P in N from b_i to b_j.

Recall that a **subtraction-free rational expression is an expression** is an expression with positive integer coefficients that can be written with the operations of addition, multiplication, and division.

$$\frac{x + z^2/y}{z^2 + 25y/x + s} = \frac{(x + z^2)(x + s)}{z^2 y + 25y^2}$$

- We can now define the **boundary measurements** M_{ij} as the specializations of the **formal boundary measurements** M_{ij}^{form}, written as subtraction-free expressions, when we assign the x_e to be the positive real weights of edges e in the network N
- What information about a planar directed network can be recovered from the collection of boundary measurements M_{ij}? How to recover this information?

Let Net_{kn} be the set of planar directed networks with k boundary sources and $n - k$ boundary sinks. Define the **boundary measurement map**

$$Meas : Net_{kn} \to Gr_{kn} \tag{1}$$

followingly. For a network $N \in Net_{kn}$ with the source set I and with the boundary measurements M_{ij}, the point $Meas(N) \in Gr_{kn}$ is given in terms of its Plücker coordinates by the conditions that $\Delta_I \neq 0$ and

$$M_{ij} = \Delta_{(I \setminus \{i\}) \cup \{j\}} / \Delta_I, \tag{2}$$

for any $i \in I$ and $j \in \bar{I}$.

More, explicitely if $I = \{i_1 < i_2 < ... < i_k\}$, then the point $Meas(N) \in Gr_{kn}$ is represented by the boundary measurement matrix $A(N) = (a_{ij}) \in Mat_{kn}$ such that

1. The submatrix $A(N)_I$ in the column set I is the identity matrix Id_k.
2. The remaining entries of $A(N)$ are $a_{rj} = (-1)^s M_{i_r,j}$, for $r \in [k]$ and $j \in \overline{I}$, where s is the number of elements of I strictly between i_r and j.

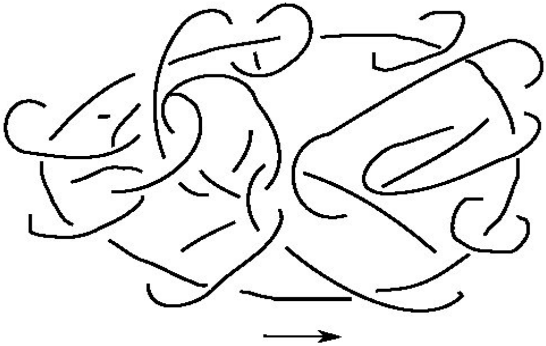
Theorem

The image of the **boundary measurement map** *Meas is exactly the totally nonnegative Grassmanian:*

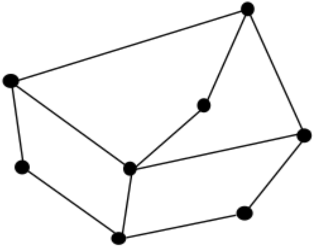
$$Meas(Net_{kn}) = Gr_{kn}^{tnn}$$

Hopf-linked rings

ALGEBRA $\times$ TOPOLOGY



Feynman path integral



Slogan for people, who will be interested in RT-paradigm

"Mathematics done in a limited way could not make any harm in any extension."
[It means when we do mathematics in a restricted way, it could not make us any harm when we do it as much as possible. We want to say that we will need to work extremely hard when we would like to solve the mathematical problems in RT-paradigm.]

Thank You for Your attention!

MERCY POR ATENSIOON! (KAHCG)

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Pictures of scientists were taken from web.