Knots and links in physics

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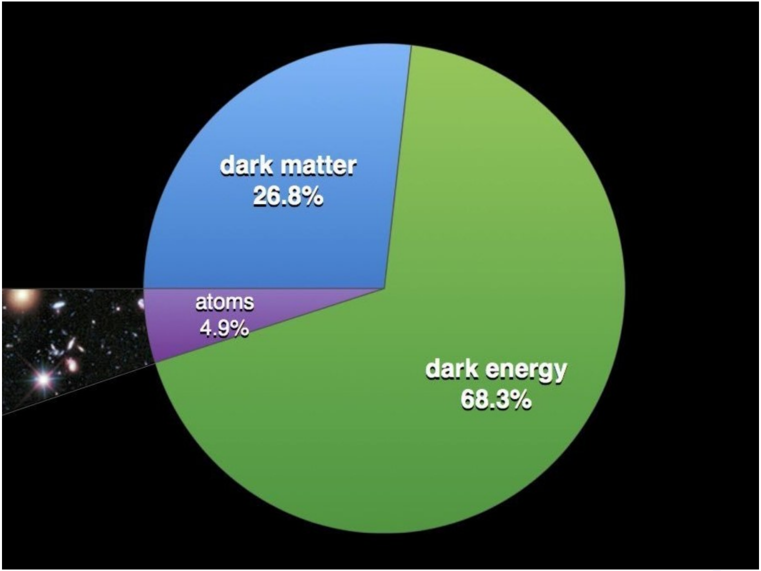
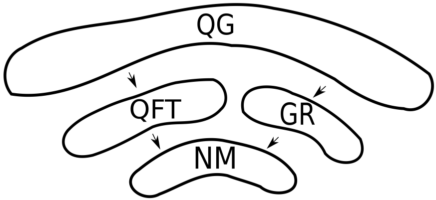
Dark energy



Classical explanation of the problem of dark energy

- add **cosmological constant** to Einstein equations
- introduce a new field: **scalar field** (phantom, quintom, quintessence)
- **modification of gravity**:
  1. modify the law of gravity at large distances
  2. build the models of gravity from higher-dimensional models

Quantum gravity and dark energy



1. string theory (origin from particle physics models)
2. loop quantum gravity (canonical quantization in general relativity)
3. causal set approach (built on 6 axioms)

Modified string theory

- we observe de Sitter space (because of the accelerated expansion)
- string theory (ST) abhors the de Sitter space
- in a generic, non-commutative generalized geometric phase space formulation of ST is possible to solve the problem of dark energy (reference D. Minic)

Loop quantum gravity and other discrete approaches

- consider microscopic structure of spacetime and its interaction with matter
- discreteness of geometry and Lorentz invariance at low energies is a key aspect of QG
- massive fields are the natural candidates for probes of spacetime discreteness

The 'friction-like' force must be proportional to **R** (**Ricci scalar**), mass m, **4-velocity** u^μ, **spin of the particle** s^μ and **time-like unit vector** ξ^μ specifying the local frame defined by the matter that curves spacetime. The formula is the following:

$$u^\mu \nabla_\mu u^\nu = \alpha \frac{m}{m_p^2} \mathrm{sign}(s \cdot \xi) \mathbf{R}\, s^\nu ,$$

where α is a dimensionless coupling.

Causal set approach

We suppose in quantum gravity that the relation

$$\Delta\Lambda\Delta V \sim \hbar$$

holds, where $\Delta\Lambda$ is a fluctuation in cosmological constant in given volume V.
The central result:

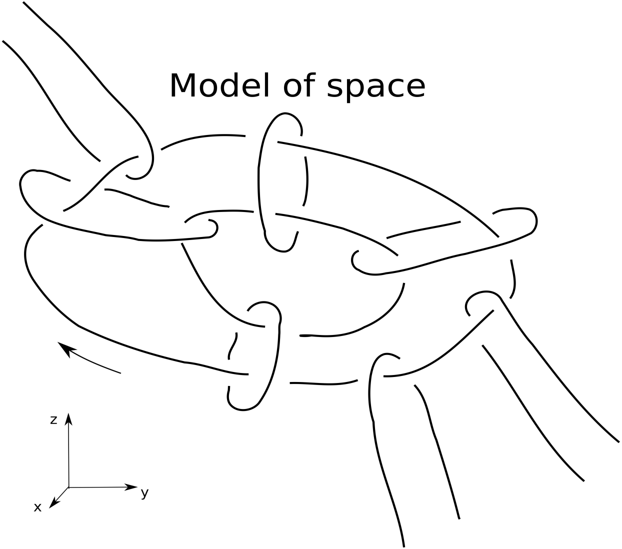
$$\Delta\Lambda \sim \frac{1}{\sqrt{V}}$$

The standard cosmological argument:

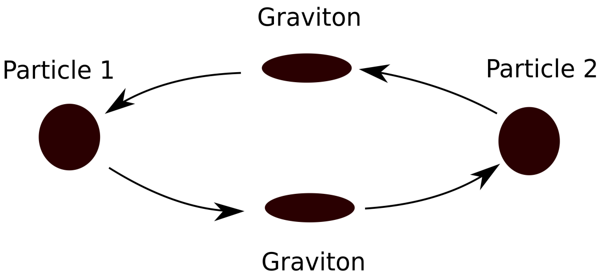
$$V \sim (H^{-1})^4 = H^{-4} \Rightarrow \Lambda \sim \frac{1}{\sqrt{V}} \sim H^2 \sim \rho_{crit}$$

It implies that Λ will be everpresent at least in $3 + 1$ dimensions.

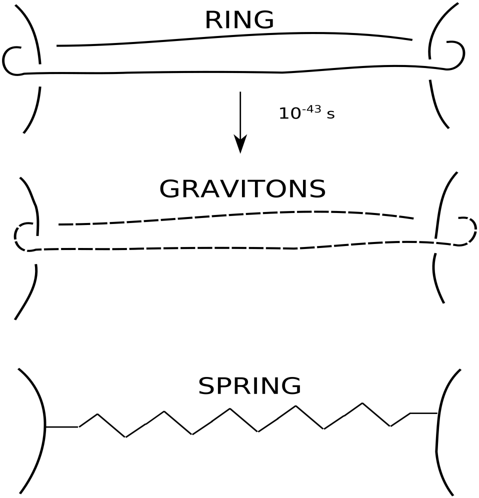
Ring paradigm



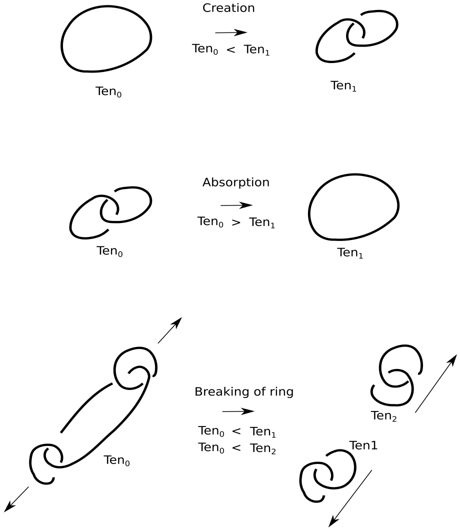
Old concept of interaction in gravity sector



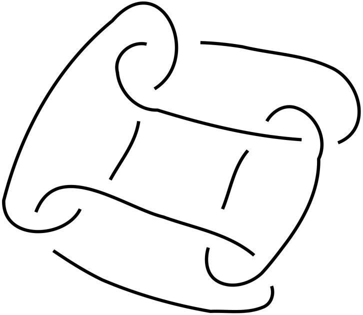
Graviton as a phonon

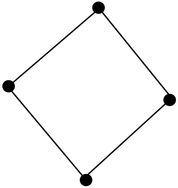


Model of dark energy in ring paradigm



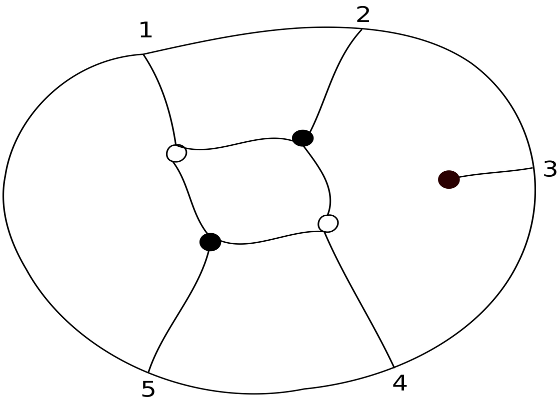
Mathematical problem: we have a finite collection of N rings S^1 in R^3, which could not touch; Give a **complete characterization of all non-homeomorphic structures**, which could be constructed from this finite collection of rings. Every two rings could be linked only once, they could not be knotted or twisted (in the case we have differentiable structure). We do not consider any Brunnian type of link in 3 and more rings.

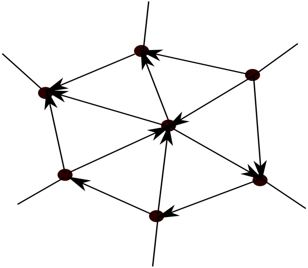




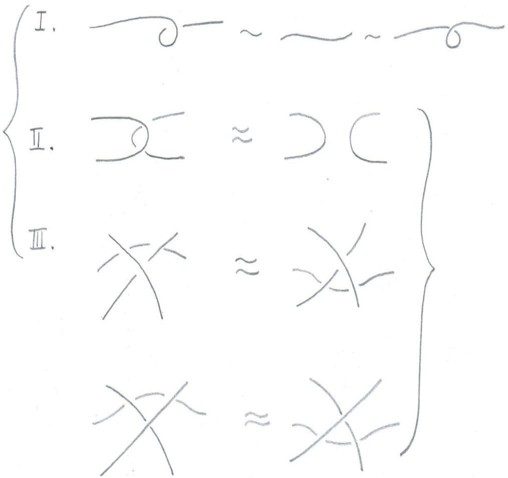
Definition

A **plabic graph** is an undirected planar graph G, which we draw inside a disk (considered modulo homotopy) with n boundary vertices on the boundary of the disk, labeled $1, \ldots, n$ in clockwise order, as well as some colored internal vertices. These internal vertices are strictly inside the disk and are each colored either black or white. Each boundary vertex i in G is incident to a single edge. If a boundary vertex is adjacent to a leaf (vertex of degree 1), we refer to that leaf as a lollipop.


1
2
3
4
5

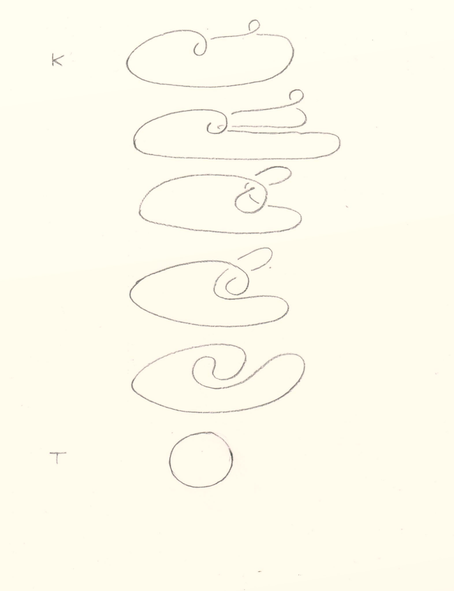


Reidemeister moves

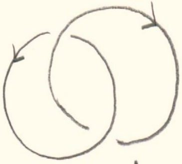


The three moves on diagrams (I, II, III) change the **graphical structure** of the diagram, but they are leaving the **topological type** of the embedding of the corresponding knot or link the same.

Exercise

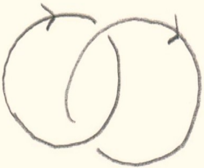


Linking number



L

$$lk(L) = \frac{1}{2}(1+1) = 1$$



L'

$$lk(L') = -1$$

Bracket polynomial

We define the **bracket polynomial** by the formula

$$\langle K \rangle = \langle K \rangle(A, B, d) = \sum_{\sigma} \langle K | \sigma \rangle d^{||\sigma||}$$

where A, B and d are commuting algebraic variables. σ runs over all the states of K.

Example

Bracket polynomial for the trefoil diagram

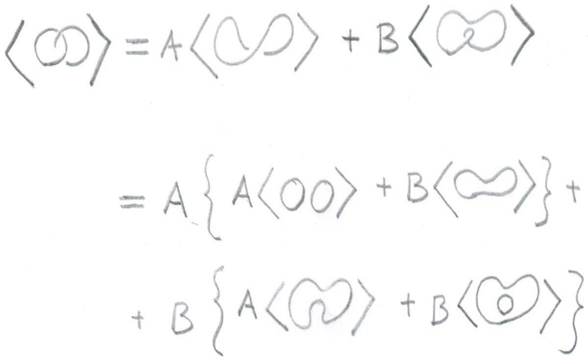
$$\langle K \rangle = A^3 d^{2-1} + A^2 B d^{1-1} + A^2 B d^{1-1} + A B^2 d^{2-1} + A^2 B d^{1-1} + \\ + A B^2 d^{2-1} + A B^2 d^{2-1} + B^3 d^{3-1}$$

$$\langle K \rangle = A^3 d^1 + 3A^2 B d^0 + 3AB^2 d^1 + B^3 d^2$$

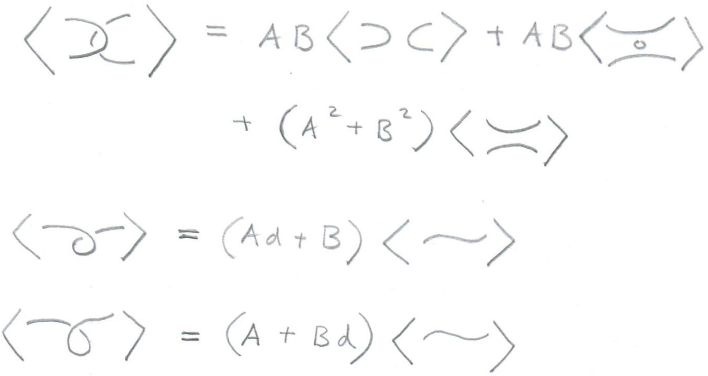
This bracket polynomial is **not** a topological invariant as it stands. We investigate the behaviour under the Reidemeister moves - and determine the conditions on A, B and d to become an invariant.

Lemma

Application of the lemma

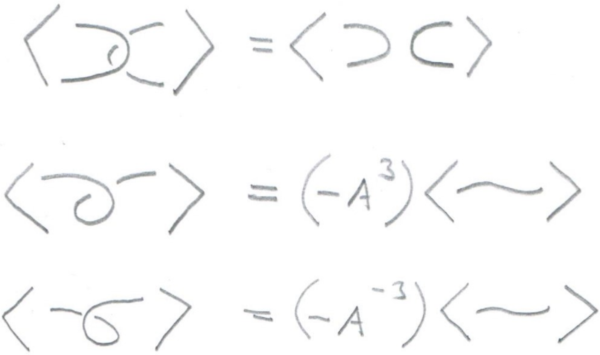


Proposition

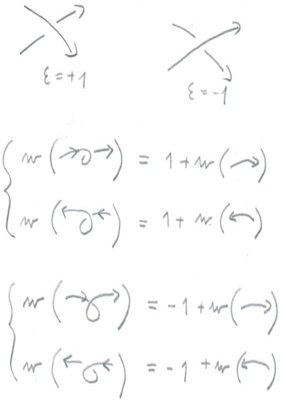


Corollary

If $B = A^{-1}$ and $d = -A^2 - A^{-2}$,



Let K be an oriented link diagram. Define the **writhe of** K, $w(K)$, by the equation $w(K) = \sum_p \epsilon(p)$ where p runs over all crossings in K and $\epsilon(p)$ is the sign of the crossing:



So, we can define a **normalized bracket**, Λ_K for oriented links K by the formula

$$\Lambda_K = (-A^3)^{-w(K)} \langle K \rangle$$

and we have the proposition

Lemma

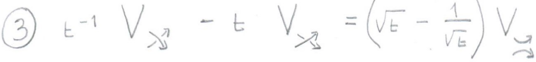
The normalized bracket polynomial Λ_K is an invariant of ambient isotopy.

Jones polynomial

The bracket polynomial gives rise to the **Jones polynomial**. The 1-variable Jones polynomial, $V_K(t)$, is a Laurent polynomial in the variable t assigned to an oriented link K. The polynomial satisfies the properties:

1. If K is ambient isotopic to K', then
$$V_K(t) = V_{K'}(t).$$

2. $V_{\bigcirc} = 1$



Theorem

Theorem

Let $\Lambda_K = (-A^3)^{-w(K)}\langle K \rangle$ *is the normalized bracket. Then*

$$\Lambda_K(t^{-\frac{1}{4}}) = V_K(t).$$

Thus the normalized bracket yields the 1-variable Jones polynomial.

Other polynomials

Convay-Alexander:

$$\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0} \qquad (1)$$

Homfly:

$$\alpha P_{K_+} - \alpha^{-1} P_{K_-} = zP_{K_0} \qquad (2)$$

Braids and Jones polynomial

The normalized bracket $\Lambda_K(A) = (-A^3)^{-w(K)}\langle K\rangle$ is a version of the original Jones polynomial by the theory of braids.
We consider a sequence of algebras A_n $(n = 2, 3, ...)$ with multiplicative generators e_1, e_2,..., e_{n-1} and relations:

1. $e_i^2 = e_i$
2. $e_i e_{i\pm1} e_i = \tau e_i$
3. $e_i e_j = e_j e_i$, $|i - j| > 2$

(τ is a scalar, commuting with all other elements.)

Jones was surprised by the analogy between the relations for the algebra A_n and the generating relations for the **n-strand Artin braid group** B_n:

1. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
2. $\sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| > 1$

for some elements $\sigma_1^{\pm 1}, ..., \sigma_{n-1}^{\pm 1}$.

He constructed a **representation** $\rho_n : B_n \to A_n$ of the Artin Braid group to the algebra A_n. The representation has the form

$$\rho_n(\sigma_i) = ae_i + b, \tag{3}$$

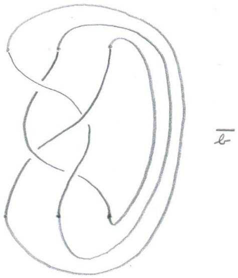
with a and b chosen appropriately.
Since A_n has a trace $tr : A_n \to \mathbb{C}[t, t^{-1}]$ one can obtain a mapping $tr \circ \rho : B_n \to \mathbb{C}[t, t^{-1}]$. After appropriate normalization this mapping is the Jones polynomial $V_K(t)$. It is an ambient isotopy invariant for oriented links.

Braids

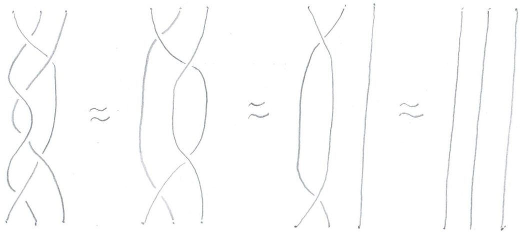
Alexander's theorem:

Theorem

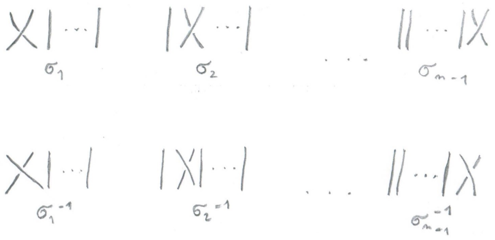
Each link in three-dimensional space is ambient isotopic to a link in the form of a closed braid.



Two braids in B_n are said to be **equivalent** if there is an ambient isotopy from b to b' which keeps the end-points fixed and does not move any strands outside the space between the top and bottom planes of the braids.



Every braid can be written as a product of the **generators** $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and their inverses $\sigma_1^{-1}, \sigma_2^{-1}, ..., \sigma_{n-1}^{-1}$. These elementary braids σ_i and σ_i^{-1} are obtained by interchanging only the i-th and $i+1$-th points the row of inputs.



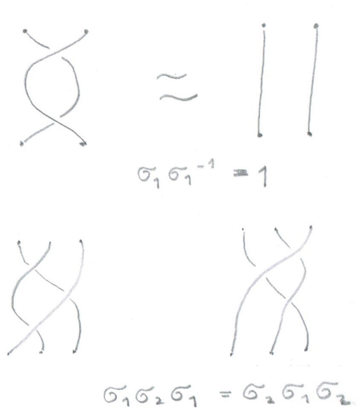
Catalog various weaving patterns:

$$= \sigma_1^{-1}\sigma_2\,\sigma_1^{-1}\sigma_2\,\sigma_1^{-1}\sigma_2$$

The braid group B_n is completely described by these generators and relations. The relations are the following:

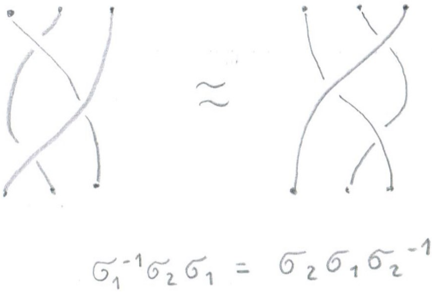
1. $\sigma_i \sigma_i^{-1} = 1, \;\; i = 1, ..., n-1$
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \;\; i = 1, ..., n-2$
3. $\sigma_i \sigma_j = \sigma_j \sigma_i, \;\; |i-j| > 1.$

The first relation is a version of the type II move, while the second relation is a type III move:



Since $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ is stated in the group B_3, we also know that $(\sigma_1 \sigma_2 \sigma_1)^{-1} = (\sigma_2 \sigma_1 \sigma_2)^{-1}$, whence $\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} = \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}$.

There are few other cases of the type III move. For example:

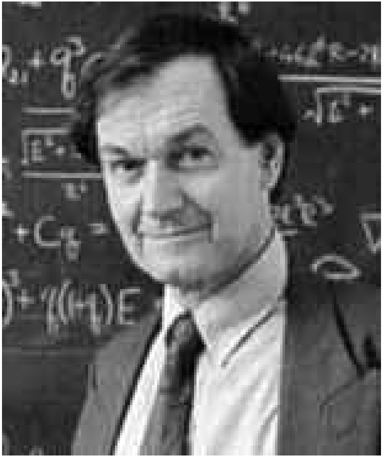


This is algebraically equivalent to the relation $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$.

Conclusion

- it is possible that the problem of dark energy is solvable only in quantum gravity
- we suggested a new approach (ring paradigm), which has advantages in cosmology
- ring paradigm is connected with very nice mathematics of knots and links

"Science and fun cannot be separated."



Some pictures were taken from web.

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